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DIFFERENCE METHODS FOR SINGULAR PERTURBATION PROBLEMS

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Monographs and Surveys in Pure and Applied Mathematics

Dedication

Dedicated to the memory of academicians
Alexandr Andreevich Samarskii and
Nikolai Sergeevich Bakhvalov

Preface

The present book is devoted to the *development of difference schemes* that *converge ε-uniformly* in the maximum norm for a representative class of *singularly perturbed problems*. It also deals with the *justification* of their *convergence*, and *surveys new directions and approaches developed recently*, which are of importance for further progress in numerical methods.

The book was intended to be an English translation of the Russian book [138] (Shishkin G.I. (1992). *Discrete Approximations of Singularly Perturbed Elliptic and Parabolic Equations.* Russian Academy of Sciences, Ural Section, Ekaterinburg (in Russian)) that was initiated by John J.H. Miller. The translation was made by Zora Uzelac, but we decided not to publish this version of the book. The very dense nature of this book, that allowed us to cover a large class of singularly perturbed boundary value problems in little space, was too difficult for most readers and also created problems in the implementation of the results. Since the appearance of the book [138], new results and ideas have appeared that are dealt with in the present book.

First, I would like to thank my teachers. My scientific interests in computational mathematics were formed and matured under the influence of the scientific schools of the Academicians of the Russian Academy of Science. A.M. Il'in, A.A. Samarskii, N.S. Bakhvalov, G.I. Marchuk and their influence led to the appearance of my second doctoral thesis. This thesis became the basis of [138], and is a continuing influence on my work.

It is with pleasure that I note the long-term and fruitful collaboration with the Irish and Dutch mathematicians in the groups of J. Miller and P. Hemker. This collaboration began in 1990, and yielded progress in the development of numerical methods for problems with boundary layers, and led to new results that were published in numerous joint papers and in two books [87] and [33].

The Russian scientists K.V. Emelianov, V.D. Liseikin, P.N. Vabishchevich, V.B. Andreev, V.F. Butuzov, A.V. Gulin, I.G. Belukhina, N.V. Kopteva, V.V. Shaidurov, B.M. Bagaev, E.D. Karepova, M.M. Lavrentiev, Jr, Yu.M. Laevsky, A.I. Zadorin, A.D. Ljashko and I.B. Badriev also influenced much of the detail of the approaches initiated in [138].

The idea to translate into English the book [138] began during my collaboration over the last dozen years with mathematicians and their students, namely, J.J.H. Miller, E. O'Riordan, A.F. Hegarty, M. Stynes, A. Ansari (Ireland), P.W. Hemker, J. Maubach, P. Wesseling (the Netherlands), P.A. Farrell (USA), F. Lisbona, C. Clavero, J.L. Gracia, J.C. Jorge (Spain), D. Creamer, Lin Pin (Singapore), and through discussions of papers (based on ideas from

[138]) on international conferences with, among others, I.P. Boglaev (New Zealand), R.E. O'Malley, R.B. Kellogg (USA), L. Tobiska, H.-G. Roos, G. Lube, T. Linß (Germany), Wang Song (Australia), L.G. Vulkov, I.A. Braianov (Bulgaria), R. Čiegis (Lithuania), and P.P. Matus (Belarus). Numerous ideas from [138] were extended and published in many papers.

My thanks especially to L.P. Shishkina, my better half, and main assistant-colleague and mathematician for participation as co-author in writing this book, for enormous scientific and technical support. She has prepared the present book including all stages: the clarification of results by numerous discussions, preparation in LaTeX, the translation, compiling the Index, and reviewing the page-proofs.

Significant assistance in the preparation of the English version of the present book, in the translation from Russian-English to idiomatic English, was made by M. Stynes (Part I, and fragments of Part II) and M. Mortell (the Preface and the Introduction) to whom I would like to express my deepest thanks.

My thanks to our assistant-colleague I.V. Tselishcheva for support in the process of preparing the book, participation in the translation of some chapters from Part II, of the Introduction, of the Survey, and many other tasks.

I am grateful for financial and material support (scientific books, computational technique) to the

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G.I. Shishkin

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Part I

Grid approximations of singular perturbation partial differential equations

Chapter 1

Introduction

1.1 The development of numerical methods for singularly perturbed problems

The wide use of computing techniques, combined with the demands of scientific and technical practices, has stimulated the development of numerical methods to a great extent, and in particular, methods for solving differential equations. The efficiency of such methods is governed by their accuracy, simplicity in computing the discrete solution and also their relative insensitivity to parameters in the problem. At present, numerical methods for solving partial differential equations, in particular, finite difference schemes, are well developed for wide classes of boundary value problems (see, for example, [79, 108, 100, 214, 91, 216]).

Among boundary value problems, a considerable class includes problems for singularly perturbed equations, i.e., differential equations whose highest-order derivatives are multiplied by a (perturbation) parameter ε. The perturbation parameter ε may take arbitrary values in the open-closed interval $(0, 1]$ (see, e.g., [211, 210, 57, 94, 62]). Solutions of singularly perturbed problems, unlike regular problems, have boundary and/or interior layers, that is, narrow subdomains specified by the parameter ε on which the solutions vary by a finite value. The derivatives of the solution in these subdomains grow without bound as ε tends to zero.

In the case of singularly perturbed problems, the use of numerical methods developed for solving regular problems leads to errors in the solution that depend on the value of the parameter ε. Errors of the numerical solution depend on the distribution of mesh points and become small only when the effective mesh-size in the layer is much less than the value of the parameter ε (see, e.g., [138, 87, 106, 33]). Such numerical methods turn out to be inapplicable for singularly perturbed problems.

Due to this, there is an interest in the development of special numerical methods where solution errors are independent of the parameter ε and defined only by the number of nodes in the meshes used, i.e., *numerical methods* (in particular, *finite difference schemes*) that *converge ε-uniformly*. When the solutions by such methods are *ε-uniformly convergent*, we will call these methods and solutions *robust* (as in [33]). At present, only several books are devoted to the *development* of numerical methods for solving singularly