METHODS FOR LINEAR AND QUADRATIC PROGRAMMING

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METHODS FOR LINEAR AND QUADRATIC PROGRAMMING

STUDIES IN MATHEMATICAL AND MANAGERIAL ECONOMICS

Editor

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The Editor hopes that the volumes of this series, all of which relate to such a young and vigorous field of research activity, will contribute to the exchange of scientific information at a truly international level.

THE EDITOR

PREFACE

This book is the outcome of a number of years of work in the area of linear and quadratic programming. Most of it is based on articles and papers written together by Andrew Whinston or by this author only.

While writing this book, I had two objectives in mind. The first one was to provide a detailed exposition of the most important methods of linear and quadratic programming which will introduce these methods to a wide variety of readers. For this purpose, numerical examples are given throughout the book. The second objective was to relate the large number of methods which exist in both linear and quadratic programming to each other. In linear programming the concepts of dual and parametric equivalence were useful and in quadratic programming that of symmetric and asymmetric variants.

Throughout the book, the treatment is in terms of methods and tableaux resulting from these methods rather than mathematical theorems and proofs, or, in other terms, the treatment is constructive rather than analytical. The main advantage of such a constructive approach is thought to be in the accessibility of the methods. Whereas an analytical approach introduces a method through a maze of theorems after which the methods appear as an afterthought, a constructive approach first states the main principles of the method, after which obstacles and implications are dealt with one by one.

A disadvantage of detailed exposition and of numerical examples is lack of conciseness. This has resulted in the limitation of the number of topics treated in this book. Hence such topics as decomposition methods for linear and quadratic programming, quadratic transportation problem and integer linear and quadratic programming are missing in this book, though most concepts on which methods for these problems are based follow rather naturally from the methods which are treated. However, inclusion of a number of these subjects would have increased the size of this book unduly.

The linear complementarity problem is treated in some detail, not only because it is an immediate generalization of linear and quadratic programming but also because it is amenable to an interesting generalization of the parametric methods which form the core of this book.

The book can be used for graduate or senior undergraduate courses in mathematical programming. In case of a one-year course, it could serve as a basis for the first half; the second half would then deal with general nonlinear programming and integer programming.

The mathematical prerequisite of the book is that the reader should have a working knowledge of elementary matrix algebra.

Thanks are due to Henri Theil, who has introduced me to the subject of quadratic programming by asking me to participate in the development of a method for quadratic programming (which can be found in chapter 12) and to Andrew Whinston, with whom I had a number of years of fruitful cooperation and who was a coauthor of a number of the articles on which this book is based.

Thanks are also due to the National Research Council of Canada for research support and to *Econometrica*, *The International Economic Review*, and *Operations Research* for allowing me to use material published in these journals for parts of chapters 6, 9 and 12.

C. VAN DE PANNE

CONTENTS

CHAPTER 1

THE SOLUTION OF LINEAR SYSTEMS

1.1. General linear systems and canonical forms

Let us consider an equation system of the following form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2, \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m. \end{aligned} \tag{1}$$

This is a system of m linear equations in n variables $x_1, \dots, x_n$. The system (1) may have no solution, one solution or a number of solutions; this depends on the coefficients of the system $a_{11}, \dots, a_{mn}$ and $b_1, \dots, b_m$.

The solutions, if any, of a general system like (1) cannot immediately be found from the system. Systems having some special form may provide more information about their solutions. Consider a system of the following form:

$$\begin{aligned} x_1 \quad + a_{1,m+1}x_{m+1} + a_{1,m+2}x_{m+2} + \dots + a_{1n}x_n &= b_1, \\ x_2 \quad + a_{2,m+1}x_{m+1} + a_{2,m+2}x_{m+2} + \dots + a_{2n}x_n &= b_2, \\ \dots \\ x_m + a_{m,m+1}x_{m+1} + a_{m,m+2}x_{m+2} + \dots + a_{mn}x_n &= b_m. \end{aligned} \tag{2}$$

One solution of this system is, for example,

$$x_1 = b_1, x_2 = b_2, \dots, x_m = b_m, x_{m+1} = x_{m+2} = \dots = x_n = 0. \tag{3}$$

Other solutions of this system may be found by choosing any values for the variables $x_{m+1}, x_{m+2}, \dots, x_n$ and computing the corresponding values for the first m x-variables. If the particular values chosen for the last $n-m$ x-variables are indicated by bars, we have for instance for the corresponding value of x_1, indicated by $\bar{x}_1$:

$$\bar{x}_1 = b_1 - a_{1,m+1}\bar{x}_{m+1} - a_{1,m+2}\bar{x}_{m+2} - \dots - a_{1n}\bar{x}_n. \tag{4}$$

In fact, any solution satisfying (2) may be obtained in this way. Hence it may be said that systems having forms like (2) give immediate access to any solution of the system.

The system (2) is said to be in *ordered canonical form*. A system is in ordered canonical form if the first variable only appears in the first equation, having a unity coefficient, the second variable only appears in the second equation, having a unity coefficient, and so on. The variables which appear in one equation only with a unity coefficient in that equation are called the *basic variables* of the system; each basic variable is connected with one equation. Hence in a system in ordered canonical form, the first variable is a basic variable and is connected with the first equation; the second variable is a basic variable and is connected with the second equation, and so on. In a system of m equations in ordered canonical form, the first m variables are the basic variables; the remaining $n-m$ variables are called the *nonbasic variables*.

Consider the following system:

$$\begin{aligned} 2x_1 + 5x_2 \qquad + x_4 &= 10, \\ 3x_1 - 4x_2 + x_3 \qquad &= 12. \end{aligned} \tag{5}$$

In this system x_4 is a basic variable in the first equation and x_3 in the second equation, but the system is not ordered. Such a system is said to be in *canonical form*. A system in canonical form can easily be brought into *ordered* canonical form by renaming and rearranging variables. For example, in the above system, x_4 may be renamed as x_1^*, x_3 as x_2^*, x_1 as x_3^*, and x_2 as x_4^*. Rearranging, we find

$$\begin{aligned} x_1^* \qquad + 2x_3^* + 5x_4^* &= 10, \\ x_2^* + 3x_3^* - 4x_4^* &= 12, \end{aligned} \tag{6}$$

which is a system in ordered canonical form. It may therefore be concluded that the difference between an ordered and a general canonical form is a rather trivial one. In the following we shall in most cases deal with general canonical forms, but in some cases it will be convenient to write such a system in ordered canonical form.

1.2. Reduction to canonical form

Two linear equation systems which have the same solutions are called *equivalent*. Because it is much easier to find solutions of systems in canonical form than solutions of general equation systems, it is desirable to find a system in canonical form which is equivalent to a given general system. First it will be shown how an equivalent system can be derived from a given system, or, which is the same, how a given system may be transformed into an equivalent one. After that, it is shown how the transformations can be used to obtain from any given system an equivalent canonical form, if this is at all possible.

The solutions of a system of equations remain the same if an equation of that system is replaced by c times that equation, where c is a nonzero constant. This can be shown as follows. Let the equation concerned be the first one of (1). This equation may be replaced by

$$c(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n) = cb. \tag{7}$$

Any solution satisfying (1) will also satisfy the modified system (1), in which the first equation has been replaced by (7). Furthermore, any solution satisfying the modified system must also satisfy the original system, since the first equation of (1) and equation (7) have the same solutions. Hence the original and the modified system must have the same solutions.

The solutions to an equation system also remain the same if an equation is replaced by the sum of that equation and a multiple of another equation of the system. Let us, for example, add to the second equation of (1) c times the first equation. We then have

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + c(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n) = b_2 + cb_1. \tag{8}$$

Any solution which satisfied (1) will satisfy (1) with the second equation replaced by (8); on the other hand, solutions satisfying (1) with (8) instead of the second equation will also satisfy (1) because, if the first equation of (1) is satisfied, the corresponding parts of (8) cancel. Hence also this operation leaves the solutions of the system unchanged.

We may therefore multiply (or divide) any equation by a nonzero constant or add to an equation a multiple of another equation without altering the solutions of the system. These two operations will be used to transform a general equation system into one in canonical form, if that is possible.

These two standard transformations are used to transform a general system into one in canonical form. This is done as follows. Suppose we want to bring the system (1) in canonical form. The system is first transformed in such a way that x_1 appears in the first equation only, and with a unit coefficient; after that, the resulting system is transformed in such a way that x_2 appears only in the second equation, with a unit coefficient, and so on, until finally a system in canonical form is obtained.

The first transformation is started by dividing the first equation by a_{11}, in order to give x_1 a unit coefficient in this equation. If a_{11} is zero, the first equation and another equation with a nonzero x_1-coefficient are interchanged. The resulting equation is

$$x_1 + a_{11}^{-1} a_{12} x_2 + a_{11}^{-1} a_{13} x_3 + \dots + a_{11}^{-1} a_{1n} x_n = a_{11}^{-1} b_1. \tag{9}$$

This may be written as

$$x_1 + b_{12} x_2 + b_{13} x_3 + \dots + b_{1n} x_n = b_{10}. \tag{10}$$

with

$$b_{12} = a_{11}^{-1} a_{12},\ b_{13} = a_{11}^{-1} a_{13},\ \dots,\ b_{1n} = a_{11}^{-1} a_{1n},\ b_{10} = a_{11}^{-1} b_1.$$

The coefficients of x_1 in the other equations are made zero by adding to these equations appropriate multiples of (9). Hence we add to the second equation $-a_{21}$ times (9). The second equation then becomes

$$(a_{22} - a_{21} a_{11}^{-1} a_{12}) x_2 + (a_{23} - a_{21} a_{11}^{-1} a_{13}) x_3 + \dots + (a_{2n} - a_{21} a_{11}^{-1} a_{1n}) x_n = b_2 - a_{21} a_{11}^{-1} b_1; \tag{11}$$

this can be written as

$$b_{22} x_2 + b_{23} x_3 + \dots + b_{2n} x_n = b_{20}, \tag{12}$$

with

$$\begin{aligned} b_{22} &= a_{22} - a_{21} a_{11}^{-1} a_{12} = a_{22} - a_{21} b_{12}, \\ b_{23} &= a_{23} - a_{21} a_{11}^{-1} a_{13} = a_{23} - a_{21} b_{13}, \\ &\dots \\ b_{2n} &= a_{2n} - a_{21} a_{11}^{-1} a_{1n} = a_{2n} - a_{21} b_{1n}, \\ b_{20} &= b_2 - a_{21} a_{11}^{-1} b_1 = b_2 - a_{21} b_{10}. \end{aligned}$$

The other equations are transformed in the same way. After the first transformation the system has the following form:

$$\begin{aligned} x_1 + b_{12} x_2 + b_{13} x_3 + \dots + b_{1n} x_n &= b_{10}, \\ b_{22} x_2 + b_{23} x_3 + \dots + b_{2n} x_n &= b_{20}, \\ &\dots \\ b_{m2} x_2 + b_{m3} x_3 + \dots + b_{mn} x_n &= b_{m0}. \end{aligned} \tag{13}$$

The transformation formulas for the coefficients, if we denote b_1 by a_{10}, b_2 by a_{20}, and so on, are

$$\begin{aligned} b_{ij} &= a_{11}^{-1} a_{ij}, & i &= 1, \\ &= a_{ij} - a_{i1} a_{11}^{-1} a_{1j} = a_{ij} - a_{i1} b_{1j}, & i &\neq 1. \end{aligned} \tag{14}$$

The second expression for $i \neq 1$ is computationally more efficient than the first one.

We have now obtained a system which is equivalent to the original system and in which x_1 is a basic variable. Now the second transformation starts which has as its objective to transform the system (13) into a system in which x_2 is also a basic variable with a unit coefficient in the second equation and zeros in the other equations. Hence we divide the second equation of (13) by b_{22}; if b_{22} is zero, the second equation and one of the remaining equations having a nonzero x_2-coefficient are interchanged. After this, suitable multiples of the resulting equation are added to the other equations of (13), the first equation included. The resulting system then has the following form

$$\begin{aligned}
x_1 \qquad + c_{13}x_3 + \ldots + c_{1n}x_n &= c_{10},\\
x_2 + c_{23}x_3 + \ldots + c_{2n}x_n &= c_{20},\\
c_{33}x_3 + \ldots + c_{3n}x_n &= c_{30},\\
\ldots\\
c_{m3}x_3 + \ldots + c_{mn}x_n &= c_{m0};
\end{aligned} \tag{15}$$

the transformation formulas are

$$\begin{aligned}
c_{ij} &= b_{22}^{-1}b_{ij}, & i &= 2,\\
&= b_{ij} - b_{i2}b_{22}^{-1}b_{2j} = b_{ij} - b_{i2}c_{2j}, & i &\neq 2.
\end{aligned} \tag{16}$$

Note that the second transformation preserves the effect of the first transformation in the sense that the coefficients of x_1 in equations other than the first stay zero. This can be seen as follows. We have

$$\begin{aligned}
c_{i1} &= b_{22}^{-1}b_{i1}, & i &= 2,\\
c_{i1} &= b_{i1} - b_{i2}b_{22}^{-1}b_{21}, & i &\neq 2,
\end{aligned} \tag{17}$$

From (13) it is obvious that $b_{i1} = 0$ for $i \neq 1$, so that c_{i1} for $i \neq 1$ must be zero. Furthermore, $c_{11} = b_{11} - b_{12}b_{22}^{-1}b_{21} = 1 - 0 = 1$.

The other transformations are performed in a similar way. If no complications occur, a system in canonical form as given in (2) may be obtained after m transformations. Before dealing with possible complications, a numerical example of the transformation of a general system into one in canonical form will be given.