DEVELOPMENTS IN MATHEMATICS 13

THEORY AND APPLICATIONS OF SPECIAL FUNCTIONS

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A Volume Dedicated to Mizan Rahman

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Developments in Mathematics

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Aims and Scope

Developments in Mathematics is a book series publishing

(i) Proceedings of conferences dealing with the latest research advances,

(ii) Research monographs, and

(iii) Contributed volumes focusing on certain areas of special interest.

Editors of conference proceedings are urged to include a few survey papers for wider appeal. Research monographs, which could be used as texts or references for graduate level courses, would also be suitable for the series. Contributed volumes are those where various authors either write papers or chapters in an organized volume devoted to a topic of special/current interest or importance. A contributed volume could deal with a classical topic that is once again in the limelight owing to new developments.

Preface

This volume, "Theory and Applications of Special Functions," is dedicated to Mizan Rahman in honoring him for the many important contributions to the theory of special functions that he has made over the years, and still continues to make. Some of the papers were presented at a special session of the American Mathematical Society Annual Meeting in Baltimore, Maryland, in January 2003 organized by Mourad Ismail.

Mizan Rahman's contributions are not only contained in his own papers, but also indirectly in other papers for which he supplied useful and often essential information. We refer to the paper on his mathematics in this volume for more information.

This paper contains some personal recollections and tries to describe Mizan Rahman's literary writings in his mother tongue, Bengali. An even more personal paper on Mizan Rahman is the letter by his sons, whom we thank for allowing us to reproduce it in this book.

The theory of special functions is very much an application driven field of mathematics. This is a very old field, dating back to the 18th century when physicists and mathematician were looking for solutions of the fundamental differential equations of mathematical physics. Since then the field has grown enormously, and this book reflects only part of the known applications.

About half of the mathematical papers in this volume deal with basic (or q-) hypergeometric series—in particular summation and transformation formulas—special functions of basic hypergeometric type, or multivariable analogs of basic hypergeometric series. This reflects the fact that basic hypergeometric series is one of the main subjects in the research of Mizan Rahman. The papers on these subjects in this volume are usually related to, or motivated by, different fields of mathematics, such as combinatorics, partition theory and representation theory. The other main subjects are hypergeometric series, and special functions of hypergeometric series, and generalities on special functions and orthogonal polynomials.

The papers in this volume on basic hypergeometric series can be subdivided into three groups: (1) papers on identities, such as integral representations, addition formulas, (bi-)orthogonality relations, for specific sets of special functions of basic hypergeometric type; (2) papers on summation and transformation formulas for single or multivariable basic hypergeometric series; and (3) papers related to combinatorics and Rogers-Ramanujan type identities. Some of the papers in this volume fall into more than just one class.

In the first group we find the two papers by Gasper and Rahman; one on q-analogs of work of Tratnik on multivariable Wilson polynomials yielding multivariable orthogonal Askey-Wilson polynomials and its limit cases and the other paper on q-analogs of multivariable biorthogonal polynomials. The paper by Ismail and R. Zhang studies the q-analog $\mathcal{E}_q$ of the exponential function, giving, amongst other things, new proofs of the addition formula and its expression as a ${}_2\varphi_1$-series. They also present new derivations of the important Nassrallah-Rahman integral, and connection coefficients for Askey-Wilson polynomials. Koornwinder gives an analytic proof of an addition formula for a three-parameter subclass of Askey-Wilson polynomials in the spirit of the Rahman-Verma addition formula for continuous q-ultraspherical polynomials. Stokman's paper simplifies previous work of Koelink and Stokman on the calculation of matrix elements of infinite dimensional quantum group representations as Askey-Wilson functions for which Rahman has supplied them with essential summation formulas. He uses integral representations for these matrix elements and shows how this can be extended to the case $|q| = 1$. Stokman's paper and Rahman's summation formulas are the motivation for Rosengren's paper using Stokman's method to extend Rahman's summation formulas. The paper by Abreu and Bustoz deals with completeness properties of Jackson's third (or the ${}_1\varphi_1$) q-Bessel function for its Fourier-Bessel expansion.

In the second group of papers on summation and transformation formulas for (multivariable) basic hypergeometric series we have the above mentioned short paper by Rosengren giving summation formulas involving bilateral sums of products of two basic hypergeometric series. Schlosser derives bilateral series from terminating ones both in the single and multivariable case. Multiple transformation formulas using the q-Pfaff-Saalscütz formula recursively are obtained by Chu. Kadell discusses various summation formulas as moments for little q-Jacobi polynomials, and extends this approach to non-terminating cases of these summation formulas.

The papers in the third group present some connections between basic hypergeometric series and, number theory and combinatorics, especially Rogers-Ramanujan type identities and the partition function. Andrews discusses two-variable analogs of the Gaussian polynomial and finite Rogers-Ramanujan type identities. In the paper by Berndt, Chan, Chan and Liaw the crank of partitions and its relation to entries in Ramanujan's notebooks are discussed. The paper by Chu also shows how the multiple transformation formulas yield multiple Rogers-Ramanujan type identities.

In this volume there are two papers that deal mainly with hypergeometric series. The paper by Stanton gives cubic and higher order transformation and summation formulas for hypergeometric series by splitting series up as a sum of r series, presenting q-analogs as well. The paper by Groenevelt, Koelink and Rosengren is solely devoted to hypergeometric series. The paper contains a summation formula where the summand involves a product of two Meixner-Pollaczek polynomials and a continuous dual Hahn polynomial. Then a Lie algebraic interpretation gives a transformation pair involving non-terminating ${}_3F_2$-series, which is proved analytically.

There are four papers that deal with aspects of the general theory for special functions and orthogonal polynomials and related subjects. Suslov's paper discusses a version of the Cauchy-Hadamard theorem giving the maximum domain of analyticity of expansions of functions into orthogonal polynomials of basic hypergeometric type. Szwarc discusses nonnegative linearization for both orthogonal polynomials and its associated polynomials, and shows the equivalence to a maximum principle for a canonically associated discrete boundary value problem. In Ruijsenaars's paper the L^2-asymptotics of orthogonal polynomials on $[-1,1]$ having a c-function expansion as the degree tends to infinity is given explicitly with an exponentially decaying error term. The paper by C. Zhang deals with summation methods involving Jacobi's theta function, which gives a summation method for divergent series. This is applied to the confluence of ${}_2\varphi_1$-series to ${}_2\varphi_0$-series, and a new q-analog of the Γ-function.

The remaining papers do not fit into the scheme given above. Using basic hypergeometric series, Berg gives direct proofs of some results on distributions for exponential functionals of Lévy processes. In particular he obtains the corresponding Laplace and Mellin transforms, which have been previously obtained by different methods from stochastic processes. Clarkson discusses polynomials that occur in relation to rational solutions of the second, third and fourth Painlevé equations, in particular the Yablonskii-Vorob'ev, (generalized) Okamoto and generalized Hermite polynomials, and he demonstrates experimentally that the zeroes of these polynomials behave in a very regular fashion. DeDeo, Martínez, Medrano, Minei, Stark and Terras study Ihara-Selberg zeta functions of Cayley graphs for the Heisenberg group over certain finite rings, and discuss a corresponding Artin L-function.

This book was prepared at the University of South Florida. Denise Marks put the book together and handled all correspondence and the galley proofs. We thank Denise for all she has done for this project. Working with Denise is always a pleasure.

We take this opportunity and thank all the speakers and participants in the American Mathematical Society Special Session, all the contributing authors, and the referees, in making this book a worthy tribute to Mizan Rahman.

Orlando, FL, and Delft,
July 2004.

Mourad E.H. Ismail
Erik Koelink

This volume is dedicated to Mizan Rahman in recognition of his contributions to special functions, q-series and orthogonal polynomials.

Contents

MIZAN RAHMAN, HIS MATHEMATICS AND LITERARY WRITINGS

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Mizan studied at the University of Dhaka where he obtained his B.Sc. degree in mathematics and physics in 1953 and his M.Sc. in applied mathematics in 1954. He received a B.A. in mathematics from Cambridge University in 1958, and a M.A. in mathematics from Cambridge University in 1963. He was a senior lecturer at University of Dhaka from 1958 until 1962. Mizan decided to go abroad for his Ph.D. He went to the University of New Brunswick in 1962 and received his Ph.D. in 1965 with a thesis on Kinetic Theory of Plasma using singular integral equations techniques. After obtaining his Ph.D., Mizan became an assistant professor at Carleton University, where he spent the rest of his career. He is currently a distinguished professor emeritus there.

In this article we mainly discuss some of Mizan's mathematical results which are the most striking and influential, at least in our opinion. Needless to say, we cannot achieve completeness since Mizan has written so many interesting papers. The reference item preceded by **CV** refer to items under "Publications" on Mizan's CV while the ones without **CV** refer to references at the end of this article.

In the early part of his research career Mizan devoted some energy to questions involving statistical distributions resulting in the papers **CV**[5], **CV**[7] and **CV**[11]. Mizan spent the academic year 1972/73 at Bedford College, of the University of London on sabbatical and worked with Mike Hoare. In an e-mail to the editors, Hoare described how the liberal arts atmosphere of Bedford, set idyllically in Regent Park was well-suited for Mizan but the down side was that Physics at Bedford was a small department and "there was little resonance in the heavily algebracisized Mathematics Department under Paul Cohen." He added "This hardly seemed to matter, since we were both outsiders from what was most fashionable at the time."

Hoare's original plan was to study a one-dimensional gas model known as the Rayleigh piston, but his collaboration with Mizan went way beyond this goal. This resulted in **CV**[9] and **CV**[12]. Another problem suggested by Hoare involved urn models which made them soon realize that the urn models they were investigating were related to birth and death processes and Jacobi and Hahn polynomials. The result of their investigations are papers **CV**[13], **CV**[17], and **CV**[18]. Some probabilistic interpretations of identities for special functions were known, but it was not an active area of research. The Hoare-Rahman papers dealt with exactly solvable models where the eigenvalues and eigenfunctions have been found explicitly. Such questions led in a very natural way to certain kernels involving the Hahn and Krawtchouk polynomials. These kernels were reproducing kernels which take nonnegative values. More general bilinear forms involving orthogonal polynomials also appeared. The question of positivity of these kernels became important and Mizan started corresponding with R. Askey who, with G. Gasper, was working on positivity questions at the time and they were very knowledgeable about these questions. Through Askey and Gasper, Mizan Rahman was attracted to the theory of special functions and eventually to q-series. He mastered the subject very quickly and started contributing regularly to the subject. Within a few years, Mizan had become a world's expert in the theory of special functions in general and q-series in particular. It is appropriate here to quote from Mike Hoare's e-mail how he described the beginning of this activity. Mike wrote "After we had done some work on ... (Rayleigh Piston) ... I happened to mention a problem which I have been worrying away at for some years. This disarmingly simple notion arose from energy transfer in chemical kinetics (the Kassell model). Reformulated as a discrete 'urn model,' it corresponds to a Markoff chain for partitioning balls in boxes in which only a subset are randomized in each event. My eigenvalue solution for the simplest continuous case in Laguerre polynomials led to probability kernels (which are) effectively the same as those seen in the formulas of Erdélyi and Kogbetliantz in the 1930's special function theory." He then added "Once Mizan's interest was stimulated, he was off and running, with the early series of abstract papers you well know." Mike Hoare echoed the feelings of those of us who collaborated with Mizan when he wrote "To see Mizan at work was an amazing experience. He seldom had to cross anything out and [in] what seemed no time at all the sheets in his characteristically meticulous script would be delivered with a modest little gesture of triumph."

The Gegenbauer addition formula for the ultraspherical polynomials was found in 1875. It says

$$C_n^{\nu}(\cos\theta\cos\varphi + z\sin\theta\sin\varphi) = \sum_{k=0}^{n} a_{k,n}(\nu)(\sin\theta)^k C_{n-k}^{\nu+k}(\cos\theta)(\sin\varphi)^k C_{n-k}^{\nu+k}(\cos\varphi) C_k^{\nu-1/2}(z), \tag{1}$$

where

$$a_{k,n}(\nu) = \frac{\Gamma(2\nu-1)}{\Gamma^2(\nu)} \frac{\Gamma^2(k+\nu)\,(n-k)!(2k+2\nu-1)}{4^{-k}\,\Gamma(n+k+2\nu)}. \tag{2}$$

The continuous q-ultraspherical polynomials first appeared in the work of L. J. Rogers from the 1890's on expansions of infinite products, which contained what later became known as the Rogers-Ramanujan identities. Their weight function and orthogonality relation were found in the late 1970's, (Askey and Ismail, 1983), (Askey and Wilson, 1985). Mizan recognized the importance of these polynomials and, in joint work with Verma, they extended the Gegenbauer addition formula to the continuous q-ultraspherical polynomials. In **CV**[48] they proved

$$\begin{aligned} C_n(z;\beta\,|\,q) = \sum_{k=0}^{n} A_{k,n}(\beta) C_{n-k}\left(\cos\theta;\beta q^k\,|\,q\right) C_{n-k}\left(\cos\varphi;\beta q^k\right) \\ \times p_k\left(z;\sqrt{\beta}\,e^{i(\theta+\varphi)},\sqrt{\beta}\,e^{i(\theta-\varphi)},\sqrt{\beta}\,e^{i(\varphi-\theta)},\sqrt{\beta}\,e^{-i(\theta+\varphi)}\,|\,q\right), \end{aligned} \tag{3}$$

where $A_{k,n}(\beta)$ are constants which are given in closed form, and the polynomial $p_n(x;a,b,c,d)$ is an Askey-Wilson polynomial. This result led to a product formula for the same polynomials. At the time the Rahman-Verma addition theorem was very surprising for two reasons. Firstly, the variables θ and φ appear in the parameters of the Askey-Wilson polynomial. Even more surprising is the fact that the terms in (3) factor in an appropriate symmetric way, since this factorization was not predicted by any structure known at the time. Only later partial explanations using representation theory of quantum groups have been given (Koelink, 1997).

Askey (Askey, 1970) raised the question of finding the domain of (α, β) within $\alpha > -1, \beta > -1$ which makes the linearization coefficients $a(m,n,k)$ in

$$P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x) = \sum_{k=|m-n|}^{m+n} a(m,n,k)P_k^{(\alpha,\beta)}(x). \tag{4}$$

nonnegative. E. Hylleraas (Hylleraas, 1962) showed that the coefficients $a(m,n,k)$ satisfy a three-term recurrence relation, and showed that the case $\alpha = \beta + 1$ leads to a closed form solution, as was the case when $\alpha = \beta$. For other (α, β) (except $\beta = -\frac{1}{2}$), the coefficients were represented as double sums, and this expression cannot be used for any of the applications the writers know except for computing a few of the coefficients. In (Gasper, 1970a) and (Gasper, 1970b), G. Gasper used the recurrence relation of Hylleraas to solve the problem of the positivity of these coefficients. Mizan started working on extending Gasper's results to the continuous q-Jacobi polynomials, where the problem is much more difficult. Rahman **CV**[27] identified $a(m,n,k)$ as a ${}_9F_8$ function and then used the ${}_9F_8$-representation to prove the nonnegativity of the linearization coefficients. Later Mizan **CV**[30] used the same technique to identify the q-analogue of $a(m,n,k)$ as a ${}_{10}\varphi_9$ and establish its nonnegativity for (α, β) in a certain subset of $(-1, \infty) \times (-1, \infty)$.

The linearization coefficients in (4) are integrals of products of three Jacobi polynomials. Din (Din, 1981) proved that

$$\int_{-1}^{1} P_m(x)P_n(x)Q_n(x)dx = 0, \quad \text{for } |m-n| < k < m+n, \tag{5}$$

where $\{P_n(x)\}$ are Legendre polynomials and $\{Q_n(x)\}$ are Legendre functions of the second kind. Askey, Koornwinder and Rahman **CV**[50] extended this to the ultraspherical polynomials. Rahman and Shah **CV**[39] summed the series, which is dual to (5), namely

$$F(\theta, \varphi, \psi) := \sum_{n=0}^{\infty}(n+1/2)P_n(\cos\theta)P_n(\cos\varphi)Q_n(\cos\psi), \tag{6}$$

$0 < \theta, \varphi, \psi < \pi$. They proved that $F(\theta, \varphi, \psi) = 0$ for $|\theta - \varphi| < \psi < \theta + \varphi \leq \pi$, but $F(\theta, \varphi, \psi) = \Delta^{1/2}$ if $\psi < |\theta - \varphi|$, or $\pi < \theta + \varphi < 2\pi$ and $\theta + \varphi + \psi < 2\pi$. On the other hand, $F(\theta, \varphi, \psi) = -\Delta^{1/2}$, if $\pi \geq \psi > \theta + \varphi$. In the above

$$\begin{aligned}\Delta := &\sin((\theta+\varphi+\psi)/2)\sin((\theta+\varphi-\psi)/2)\\ &\times \sin((\theta-\varphi+\psi)/2)\sin((\varphi+\psi-\theta)/2).\end{aligned} \tag{7}$$