Number Systems
Structure and Properties

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Scott, Foresman and Company

Library of Congress Catalog Card Number 73-78544

Regional offices of Scott, Foresman and Company are located in Atlanta, Dallas, Glenview, Palo Alto, Oakland, N. J., and London, England

Number Systems

To Betty²

Preface

Mathematics for the students of today, who will be adults in the specialized society of the future, must be more than arithmetic and the rudiments of geometry, probability, and statistics. To function effectively as educated persons they not only need to know arithmetic but must understand and appreciate the structure of mathematics and the number systems. By knowing mathematical structures and systems, students will be able to function more effectively in every system they encounter from a simple "clock-arithmetic" group to the rational number system or Boolean algebra. Once they are able to recognize properties of operations and relations common to the number systems, they will be able to utilize this knowledge and the methods of obtaining it to apply such concepts in new situations. Furthermore, when structure and systems are understood, the interrelated ideas can be used as sources for the reinforcement of skills and applications and as the basis for conjectures and discovery.

To assist the achievement of such a goal in some small way, this book on the structure and properties of the real number system was prepared for the education of teachers and liberal arts majors. The experience is recommended by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America, the Cambridge Conference committee, and numerous leaders in mathematics education.

The topics, vocabulary, and problems coincide to a considerable extent with elementary mathematics, but the sophistication and college-level point of view is appropriate for teachers who are developing an understanding of the fundamental notions behind what they teach. The book contains principles, general properties, and procedures for discovering and developing new ideas that may be a source for innovative teaching techniques, but it is void of "how-to-teach" suggestions. Thus the book does not contain elementary mathematics or methods but rather some mathematics for teachers of elementary mathematics at a level and depth appropriate for their preparation or development.

The structure and properties of the real number system are presented with a reasonably rigorous development of the system of integers and the system of rational numbers and a thorough discussion of the system of whole numbers and the system of real numbers. The developments incorporate the number line with vectors, the closure property, and the existence of solutions of equations to motivate and justify definitions and to provide reasons for proofs. The properties of the operations and relations of each number system are emphasized. Then each property is stated as an abstraction and summarization of these basic and frequently encountered concepts.

As often as feasible, new topics are introduced with intuitively reasonable presentations. Once definitions, theorems, or observations have been motivated, statements are formalized, verified, and utilized. Illustrative examples appear frequently, and an abundant supply of graded exercises accompany each section. Questions below the sign ✳✳✳ are somewhat difficult, and those further identified by the sign * may take considerable thought. Answers to selected exercises appear in the back of the book as a further assistance to the student. A supplementary booklet is available, which contains chapter tests and answers to all exercises, including proofs.

In Chapter 1, Numbers and Statements, the groundwork is laid intuitively for the study. We assume the properties of the system of whole numbers; the rest is our handiwork. Logic, sets, graphs, variables, and vectors are presented as necessary topics related to the study. The properties of mathematical operations and relations recognized for the system of whole numbers are stated in general and noted, in some cases, as applicable to sets and to other systems of numbers. The first theorems do not appear in the book until §1.10. In subsequent chapters students will participate in and be responsible for proofs.

Chapters 2 and 3 are developments of the system of integers and the system of rational numbers respectively. From these developments students should obtain an understanding of the importance of some properties and relations which apply in different systems, should begin to appreciate the proof of a statement, and should find renewed meaning for previously memorized manipulations and arithmetic procedures.

Chapter 4, The System of Real Numbers, utilizes the number line, decimal expressions, and decimal generator sequences to make plausible the statements concerning the real number system. Additional topics include powers, radicals, absolute value, some iterative procedures appropriate to digital computer programming, and an introduction to trigonometry.

We wish to express our gratitude to our students, who have given us insight and direction in the development of the notes, to our colleagues who have made contributions of advice and encouragement, and to our families who have tolerated our concentration. Specific thanks are due Scott, Foresman and Company and its editorial staff, and Betty Pettofrezzo, Karen Prather, and Flora Finetti, who helped in the preparation of the manuscript.

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Chapter 1

NUMBERS AND STATEMENTS

1.1 Logic

In mathematics we make statements that we know are *true*; we also make statements that are *false*. However, we do not make statements that are both true and false. For example, these statements are true:

> Two plus two is four.
> Five is greater than three.
> A triangle has three sides.

These statements are false:

> Two plus two is not four.
> Five is not greater than three.
> A triangle is a quadrilateral.

Based on prior experience we decide whether a statement is true or false.

Each of the statements "Five is greater than three" and "Five is not greater than three" is the *negation* of the other.

> If a statement is true, then its negation is false; if a statement is false, then its negation is true.

Neither the statement "Mary has blond hair" nor the statement "Mary has red hair" is the negation of the other.(Why?)

EXAMPLE 1. Write the negation of each of the statements:

(a) An apple is an apple.
(b) The United States of America is not in the Western Hemisphere.
(c) The moon is made of green cheese.
(d) Three plus four is not five.

(a) An apple is not an apple.
(b) The United States of America is in the Western Hemisphere.
(c) The moon is not made of green cheese.
(d) Three plus four is five.

Each of the statements considered thus far is an example of a *simple statement.* A simple statement expresses a single idea. Two simple statements can be combined by the use of the word "and" to form a *compound statement* called a *conjunction.* For example, these compound statements are conjunctions:

Two plus two is four *and* a triangle has three sides.
Two plus two is four *and* five is not greater than three.

In mathematics, as in logic, it is important to determine whether a given compound statement is true or false. The truth of the compound statement

$$\text{“}\underbrace{\text{Two plus two is four}}_{\text{1st simple statement}}\ \textit{and}\ \underbrace{\text{a triangle has three sides}}_{\text{2nd simple statement}}\text{”}$$

depends on the truth of *both* simple statements.

A conjunction is true when both simple statements are true; otherwise the conjunction is false.

The conjunction

"Two plus two is four *and* five is not greater than three"

is false since the second simple statement is false.

Two simple statements can be combined by the use of the word "or" to form a compound statement called a *disjunction.* For example, these compound statements are disjunctions:

Five is greater than three *or* two plus two is not four.
Five is greater than three *or* a triangle has three sides.

The truth of the compound statement

$$\text{“}\underbrace{\text{Five is greater than three}}_{\text{1st simple statement}}\ \textit{or}\ \underbrace{\text{two plus two is not four}}_{\text{2nd simple statement}}\text{”}$$

depends on the truth of *either* simple statement.

A disjunction is true when at least one of the simple statements is true; otherwise, the disjunction is false.

The disjunction

"Five is greater than three *or* a triangle has three sides"

is true since at least one of the simple statements is true. Indeed, both simple statements are true. The disjunction

"Two plus two is not four *or* a triangle is a quadrilateral"

is false since both simple statements are false.

EXAMPLE 2. Use your knowledge of arithmetic to determine whether each compound statement is true or false:

(a) $3 + 2 = 5$ *and* $3 + 1 > 2$;
(b) $3 + 0 = 0$ *and* $5 - 2 = 3$;
(c) $2 + 3 = 6$ *or* $2 \times 3 = 6$;
(d) $1 + 1 < 2$ *or* $1 + 1 > 2$.

(a) The conjunction "$3 + 2 = 5$ *and* $3 + 1 > 2$" is true since both simple statements are true;
(b) the conjunction "$3 + 0 = 0$ *and* $5 - 2 = 3$" is false since the first simple statement "$3 + 0 = 0$" is false;
(c) the disjunction "$2 + 3 = 6$ *or* $2 \times 3 = 6$" is true since at least one simple statement (the second simple statement "$2 \times 3 = 6$") is true;
(d) the disjunction "$1 + 1 < 2$ *or* $1 + 1 > 2$" is false since both simple statements are false.

The *truth value* (whether the statement is true or false) of a compound statement can be summarized by means of a *truth table.* Let p and q represent simple statements. Then Table 1.1 represents the truth table for the conjunction "p *and* q," where T means "true" and F means "false".

TRUTH TABLE FOR A CONJUNCTION

p	q	p *and* q
T	*T*	*T*
T	*F*	*F*
F	*T*	*F*
F	*F*	*F*

TABLE 1.1

Table 1.2 represents the truth table for the disjunction "*p or q*".

TRUTH TABLE FOR A DISJUNCTION

p	q	p *or* q
T	T	T
T	F	T
F	T	T
F	F	F

TABLE 1.2

Let "*not p*" represent the negation of the simple statement p. Then Table 1.3 represents the truth table for the negation "*not p*".

TRUTH TABLE FOR A NEGATION

p	*not* p
T	F
F	T

TABLE 1.3

EXERCISES

Use your common knowledge to determine whether each compound statement is true or false:

1. There are seven days in each week and twelve inches is equivalent to one foot.
2. There are 30 days in September and there are 31 days in June.
3. Germany is in South America or Brazil is in Europe.
4. Chicago is in Illinois and Miami Beach is in Wyoming.
5. Beethoven was a composer or George Washington was President of the United States.
6. Some roses are red and some roses are yellow.
7. You are taller than yourself or you are shorter than yourself.
8. St. Louis is east of New York or Boston is north of New York.

Use your knowledge of arithmetic to determine whether each compound statement is true or false:

9. $9 + 7 = 15$ and $9 + 8 = 16$.
10. $3 + 0 = 3$ and $5 + 7 = 7 + 5$.

11. $3 > 2$ and $3 < 8$.

12. $3 > 2$ or $3 > 8$.

13. $3(2 + 5) = 6 + 15$ and $3 + (2 + 1) = (3 + 2) + 1$.

14. $6 \neq 6$ or $3 + 1 = 1 + 3$.

15. $7 - 11 = 4$ or $4 - 11 = 7$.

16. $6 + 2 > 6 + 1$ and $8 \times 2 > 5 \times 2$.

* * *

In Exercises 17 and 18 complete each sentence:

17. A conjunction in which one simple statement is the negation of the other simple statement is ____.

18. A disjunction in which one simple statement is the negation of the other simple statement is ____.

19. Summarize the truth values of the compound statement "Not p *or* q" by means of a truth table.

20. Summarize the truth values of the compound statement "Not p *and* not q" by means of a truth table.

In Exercises 21 and 22 fill the blanks in each truth table:

21.

p	*q*	*p and q*
T	*F*	
	T	*F*
T		*T*

22.

p	*q*	*p or q*
	F	*T*
F		*F*
F	*T*	

***23.** Let p, q, and r represent simple statements. Complete the truth table:

p	*q*	*r*	(*p and q*) *or r*	*p and* (*q or r*)
T	*T*	*T*		
T	*T*	*F*		
T	*F*	*T*		
T	*F*	*F*		
F	*T*	*T*		
F	*T*	*F*		
F	*F*	*T*		
F	*F*	*F*		

1.2 Implication

An important type of compound statement in mathematics is the *if–then statement,* called an *implication.* For example, these compound statements are implications:

If $2 + 3 = 5$, *then* $2 + 4 = 6$.

If five is greater than two, *then* two is greater than three.

If a triangle has three sides, *then* it is a quadrilateral.

If $2 = 1$, *then* $3 + 2 = 3 + 1$.

In the implication "If p, *then* q" the simple statement p is called the *hypothesis*; the simple statement q is called the *conclusion.*

In order to determine the truth values of an implication, consider the following situation. The Boston Celtics are to play the San Francisco Warriors in basketball. The following compound statements have the same meaning:

The Celtics will win *or* the Warriors will win.

If the Celtics do not win, *then* the Warriors will win.

Hence we would expect these compound statements to have the same truth values. Let p represent the simple statement "The Celtics do not win"; let q represent the simple statement "The Warriors will win." Thus we expect the compound statements

"Not p *or* q" and "*If* p, *then* q"

to have the same truth values. The truth values of the implication "If p, then q" are defined to be the same as the truth values of the compound statement "Not p *or* q". Then Table 1.4 represents the truth table for the implication "*If* p, *then* q".

TRUTH TABLE FOR AN IMPLICATION

p	q	Not p	Not p *or* q	*If* p, *then* q
T	*T*	*F*	*T*	*T*
T	*F*	*F*	*F*	*F*
F	*T*	*T*	*T*	*T*
F	*F*	*T*	*T*	*T*

TABLE 1.4

An implication is false when the hypothesis is true and the conclusion is false; otherwise, the implication is true.

EXAMPLE 1. Use your knowledge of arithmetic and the rule above to determine the truth value of each statement:

(a) *If* $2 + 2 = 4$, *then* $2 + 3 = 5$.
(b) *If* $5 - 3 = 2$, *then* $3 - 5 = 2$.
(c) *If* $2 = 1$, *then* $3 + 2 = 3 + 1$.

(a) The hypothesis "$2 + 2 = 4$" is true and the conclusion "$2 + 3 = 5$" is true. Hence the implication is true.
(b) The hypothesis "$5 - 3 = 2$" is true and the conclusion "$3 - 5 = 2$" is false. Hence the implication is false.
(c) Since the hypothesis "$2 = 1$" is false, the implication is true. (Note that it is unnecessary to determine the truth value of the conclusion in this case.)

EXAMPLE 2. Using a Gregorian calendar, discuss the truth value of the statement,

"*If* today is December 25, *then* today is Christmas day."

Since this statement could be read at different times of the year, we consider what could happen. Suppose this is the day after December 24. The hypothesis would be true and the conclusion would be true. Hence the implication would be true. Suppose this is September 19 or any other day except the day after December 24. The conclusion would be false, but the hypothesis would also be false. Hence the implication would be true.

Let p and q represent simple statements. Each of the implications "*If* p, *then* q" and "*If* q, *then* p" is called the *converse* of the other. For example:

Implication	Converse of the Implication
If $3 + 4 = 7$, then $3 + 5 = 8$.	If $3 + 5 = 8$, then $3 + 4 = 7$.
If today is Monday, then tomorrow is Tuesday.	If tomorrow is Tuesday, then today is Monday.
If a geometric figure is a square, then it is a rectangle.	If a geometric figure is a rectangle, then it is a square.
If Alice passes the algebra course, then she gets an A.	If Alice gets an A, then she passes the algebra course.

From the examples above, note that an implication may be true while its converse is false; an implication may be false while its converse is true.

EXAMPLE 3. Let p and q represent simple statements. Show that the truth values of an implication "*If* p, *then* q" and its converse "*If* q, *then* p" are not the same.

p	q	*If* p, *then* q	*If* q, *then* p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

TABLE 1.5