Karl E. Gustafson

Introduction to

PARTIAL DIFFERENTIAL EQUATIONS AND HILBERT SPACE METHODS

Second Edition

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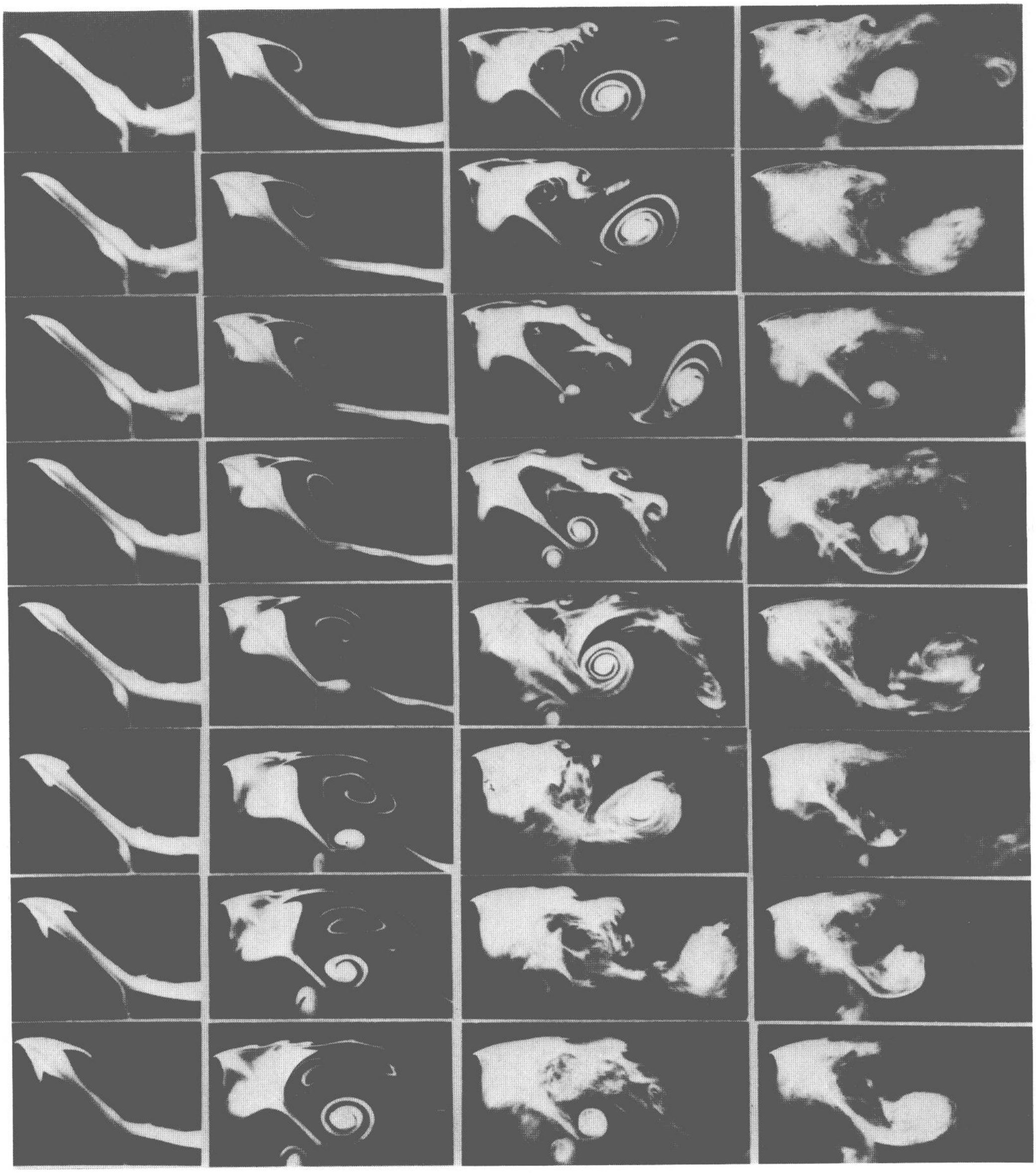

Vortex Splitting in Constant Accelerating Flow past a NACA 0015 Airfoil at angle of attack $\alpha = 40°$ and Reynolds Number Re = 1000. Courtesy P. Freymuth, *Prog Aerospace Sci*. 22 (1985). Numerical solution of partial differential equations can now simulate such vortex dynamics. See pp. 331–332.

To my children
Amy and Garth

PREFACE

One will find that the content of the first edition has been considerably enlarged by the added Chapter 3. This contains: (A) an expanded treatment of first-order systems, including elements of the theories of self-similar solutions and the more modern use of local transformation groups; (B) a short introduction to computational methods, including basics of each of the finite difference, finite element, and finite spectral schemes; and (C) a presentation of aspects of topical research on the partial differential equations of fluid dynamics, including current studies, both theoretical and computational, of fluid vortex motion and turbulence.

The reviews of the first edition, both private and public,* requested some additional examples, explanations, and exercises for the student. Accordingly, straightforward practice exercises have been inserted after the sections of Chapter 1. For expository convenience I have implemented a considerable number of examples, explanations, and exercises via a device of "Pauses" inserted twice in each of Chapters 1 and 2 at points where I found my students indeed could use a pause to recollect and practice their thoughts. Problems 1.9.9.9 and 2.9.9.9 at the ends of Chapters 1 and 2 have also been converted to provide a third set of additional exercises. Among other changes, further examples and exercises have been placed in enlarged sections on classification, characteristics, maximum principles, and numerical methods. The Answers Section at the end of the book has been significantly expanded.

I would like to take this opportunity to thank all students in my partial differential equations courses here for helping me, with their enthusiasm, ideas, and patience. I hope that this second edition will be at once easier for the beginner and more interesting for the expert. Finally, I have tried to keep the book still of manageable size.

KARL GUSTAFSON

*Principally:
Zentralblatt fur Mathematik 434:35001 (1981), P. Seydoux. *Mathematical Reviews* 8lk: 35003 (1981), R. MacCamy. *SIAM Review* 25: 110 (1983), J. Synawiec.

PREFACE TO THE FIRST EDITION

For some time now there has been a need* for a satisfactory introduction to partial differential equations and Hilbert space methods at the undergraduate level. For curricula purposes the treatment should permit both a reasonably complete one-semester course and also a somewhat more extensive full-year course. The central topics should be

Partial Differential Equations
Fourier Series and Hilbert Space
Elements of Applied Mathematics

developed according to their natural interconnections.

The first objective of this book, which derives from courses in partial differential equations taught by the author during the last ten years at the University of Minnesota, the Ecole Polytechnique Fédérale de Lausanne, and the University of Colorado, is to meet that need.

The second objective is to expose the rudiments of the subject to graduate students in mathematics and to other interested scientists. Partial differential equations play a basic role not only in applied and parts of pure mathematics, but also in physics, engineering, and other sciences, and historically motivate much of the development of mathematical analysis.

A third objective is to avoid the pitfall of becoming too involved mathematically in related matters, such as complex analysis or functional analysis. We have, however, purposely introduced some of the elementary concepts of functional analysis and spectral theory as they relate to partial differential equations. This has been done primarily in the contexts of Hilbert space and Fourier series. It has been the author's experience that the student can become quiet interested in the interplay between partial differential operators and the corresponding spectral theory as it arises naturally, for example, in physics and other applications. Moreover, a brief controlled exposure to certain important elements of functional analysis at the

*As delineated, for example, in the following studies:
CUPM (Committee on the Undergraduate Program in Mathematics).
A General Curriculum in Mathematics for College, A Report to the Mathematical Association of America (1965).
A Compendium of CUPM Recommendations: Studies, Discussions, and Recommendations by the Committee (1975).
COSRIMS (Committee on the Support of Research in the Mathematical Sciences).
The Mathematical Sciences: A Report (NAS-NRC) (1968).
AMTRAC (Committee on Applied Mathematics Training).
The Role of Applications in the Underground Mathematics Curriculum (NAS) (1979).
Notices of the American Mathematical Society (November, 1977).
NAS-NRC Committee on Applied Mathematical Training.
S. Smale, *The Major in Mathematics*.

undergraduate level is probably advisable to students going on to graduate work in mathematics, engineering, physics, or related sciences.

This book has grown out of class notes developed by the author specifically for a full-year senior-graduate course given at the University of Colorado the last three years. The enrollments in this class averaged about 40 students for the first semester, 20 to 30 students for the second semester. The breakdown has been roughly even between undergraduate and graduate students, coming mostly from mathematics, physics, and engineering. A freshman, several sophomores, and, on the other end, Ph.D. candidates in mathematics and in related fields have completed the course. There has been no great difficulty in accommodating this wide spectrum of students (but it has been advisable to treat the undergraduate and graduate students separately with regard to examination scores).

The prerequisite is calculus to the level where elementary systems of ordinary differential equations have been treated. Advanced calculus and physics are helpful but not required; the relevant facts needed are explained herein. Further formal prerequisites should be avoided, inasmuch as the course has been quite attractive to both budding young mathematicians and other scientists wishing an early exposure to mathematics as it relates to the real world, and to graduate students who wish to fill in that gap in their education. For applied mathematics and engineering majors the course offers an acceptable alternative to a year course in advanced calculus. For mathematics majors the course may be taken simultaneously with or immediately after the advanced calculus course.

Chapter 1 contains material sufficient for a one-semester course, and presents a full introduction to the subject of partial differential equations and Fourier series as related to applied mathematics. Chapter 2 begins with a more comprehensive look at the principal method for solving partial differential equations, namely, that of separation of variables, and then more fully develops that approach in the contexts of Hilbert space and numerical methods. A number of graduate students and even a few undergraduate students with some prior knowledge have successfully entered the course directly into the second semester.

A look at the Contents reveals that each of the two chapters is arranged as follows: Sections 1–8, and then Section 9.1–9.9. Sections 1 through 8 should be covered in that order, with the extent of coverage of Sections 7 and 8 somewhat optional and at the discretion of the instructor. Each of these eight sections contains problems specifically designed to firm up and add to the student's understanding of the concepts and techniques discussed in that section. Section 9 provides a "second look" at the previous topics and is broken into nine "discussion-problems." Problems 9.1 through 9.8 each augments in some way the previous same-numbered section, and each contains additional exercises for the student. Problem 9.9 ends the chapter with some "confirmation" exercises.

The instructor will notice some freedom on his or her part in choosing whether or not to treat the additional related material in Section 9 immediately after the given section. The choice will also depend on taste and classroom reaction. Our preference based on experience has been to move quickly into the basics of the subject, leaving for later discussions the further refinement of concepts and details. But as we have just indicated, the book has been intentionally written in a manner providing ample flexibility to fit alternate approaches to instruction and to learning.

Section 1.8 (Elements of bifurcation theory) and Section 2.8 (Elements of scattering theory) introduce the student to two important areas of applied mathematics that have been and currently are of considerable scientific interest. Bifurcation theory is a branch of the study of nonlinear differential equations, especially as they occur in practice in mechanics and elsewhere. Scattering theory, which is concerned with the understanding of the dynamics of interacting systems, has been and will continue to be one of the cornerstones of science.

Finally, it is hoped that this book will also be of interest to the scientific layperson who wishes to combine an appreciation of the essentials of the subject with some pleasant evening's browsings.

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GUIDE TO USE

For the advanced student or interested colleague, the book has apparently served well. For the beginning student, three acceptable introductory routes have been the following:

1. Strictly Undergraduate (e.g., senior) One-Semester Course
   Sections 1.1–1.5 inclusive: this defines the subject and provides all essential concepts.
   Problems 1.9.1–1.9.3 inclusive: these provide more detail on classification and maximum principles.
   Parts of Section 1.6, Appendix A.1, and the First and Second Pauses: further material on the divergence theorem, first-order systems, and additional practice on separation of variables, Green's functions, and variational methods.
2. Mixed Undergraduate-Graduate (e.g., first year) Two-Semester Course
   The above, plus Sections 2.1–2.2 and Problem 1.9.6: the latter provide additional rigor for the method of separation of variables.
   Portions of Section 2.3–2.5 and the Third Pause: more information about Fourier series, eigenfunction expansions, Green's Functions, and the d'Alembert solution to the wave equation.
   Section 2.6, the Fourth Pause, and Appendix B.1–B.3: as time permits, an introduction to numerical methods.
3. Graduate (e.g., second year) Courses
   First Semester, all of Chapter 1: this provides all basics, plus discussions of physical and theoretical considerations.
   Second Semester, all of Chapter 2: after a review of the special functions and convergence proofs of the method of separation of variables, theoretical connections are established to the notion of Hilbert space, the special case of Sturm–Liouville equations, variational principles and numerical methods, with a number of more advanced topics (e.g., scattering theory, nonlinear wave equations) also available.
   Third Semester, all of Chapter 3: introduction to group theoretic, computational, and physical methods for nonlinear partial differential equations.

The second edition contains (counting parts) over 600 Problems and Exercises, with answers provided or indicated for more than two-thirds of them. Especially in the initial portion of the book, the Exercises have been included to facilitate valuable easy or routine practice. The Problems, on the other hand, vary from easy to extensive. To enable more in-depth treatments of certain topics at the instructor's discretion, classic references and important recent journal articles have been cited throughout.

A NOTE TO THE READER

We have followed a pedagogical style of . . . *once* . . . *twice* . . . *and then, again,* . . . because that is how the course evolved.

One may, if one likes, justify this in retrospect in terms of the old rules of learning: attention, association, and repetition. The first encounter must be brisk and must catch the attention; the second should better associate the connections and the more general picture; the third should emphasize and repeat important features and details.

And as in all learning, one must have faith. One should not fall just because one stumbles over a detail or two at the beginning.

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