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Theory of Point Estimation

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Theory of
Point Estimation

A WILEY PUBLICATION IN MATHEMATICAL STATISTICS

To Stephen, Barbara, Fia

Preface

This book is concerned with point estimation in Euclidean sample spaces. The first four chapters deal with exact (small-sample) theory, and their approach and organization parallel those of the companion volume, *Testing Statistical Hypotheses* (TSH). Optimal estimators are derived according to criteria such as unbiasedness, equivariance, and minimaxity, and the material is organized around these criteria. The principal applications are to exponential and group families, and the systematic discussion of the rich body of (relatively simple) statistical problems that fall under these headings constitutes a second major theme of the book.

A theory of much wider applicability is obtained by adopting a large-sample approach. The last two chapters are therefore devoted to large-sample theory, with Chapter 5 providing a fairly elementary introduction to asymptotic concepts and tools. Chapter 6 establishes the asymptotic efficiency, in sufficiently regular cases, of maximum likelihood and related estimators, and of Bayes estimators, and presents a brief introduction to the local asymptotic optimality theory of Hajek and LeCam. Even in these two chapters, however, attention is restricted to Euclidean sample spaces, so that estimation in sequential analysis, stochastic processes, and function spaces, in particular, is not covered.

The text is supplemented by numerous problems. These and references to the literature are collected at the end of each chapter. The literature, particularly when applications are included, is so enormous and spread over the journals of so many countries and so many specialties that complete coverage did not seem feasible. The result is a somewhat inconsistent coverage which, in part, reflects my personal interests and experience.

It is assumed throughout that the reader has a good knowledge of calculus and linear algebra. Most of the book can be read without more advanced mathematics (including the sketch of measure theory which is presented in Section 1.2 for the sake of completeness) if the following conventions are accepted.

1. A central concept is that of an integral such as $\int f\,dP$ or $\int f\,d\mu$. This covers both the discrete and continuous case. In the discrete case $\int f\,dP$ becomes $\Sigma f(x_i)P(x_i)$ where $P(x_i) = P(X = x_i)$ and $\int f\,d\mu$ becomes $\Sigma f(x_i)$. In the continuous case, $\int f\,dP$ and $\int f\,d\mu$ become, respectively, $\int f(x)p(x)\,dx$ and $\int f(x)\,dx$. Little is lost (except a unified notation and some generality) by always making these substitutions.

2. When specifying a probability distribution P, it is necessary to specify not only the sample space $\mathcal{X}$, but also the class $\mathcal{A}$ of sets over which P is to be defined. In nearly all examples $\mathcal{X}$ will be a Euclidean space and $\mathcal{A}$ a large class of sets, the so-called Borel sets, which in particular includes all open and closed sets. The references to $\mathcal{A}$ can be ignored with practically no loss in the understanding of the statistical aspects.

A forerunner of this book appeared in 1950 in the form of mimeographed lecture notes taken by Colin Blyth during a course I taught at Berkeley; they subsequently provided a text for the course until the stencils gave out. Some sections were later updated by Michael Stuart and Fritz Scholz. Throughout the process of converting this material into a book, I greatly benefited from the support and advice of my wife, Juliet Shaffer. Parts of the manuscript were read by Rudy Beran, Peter Bickel, Colin Blyth, Larry Brown, Fritz Scholz, and Geoff Watson, all of whom suggested many improvements. Sections 6.7 and 6.8 are based on material provided by Peter Bickel and Chuck Stone, respectively. Very special thanks are due to Wei-Yin Loh, who carefully read the complete manuscript at its various stages and checked all the problems. His work led to the correction of innumerable errors and to many other improvements. Finally, I should like to thank Ruth Suzuki for her typing, which by now is legendary, and Sheila Gerber for her expert typing of many last-minute additions and corrections.

E. L. LEHMANN

Berkeley, California,
March 1983

Comments for Instructors

The two companion volumes, *Testing Statistical Hypotheses* (TSH) and *Theory of Point Estimation* (TPE), between them provide an introduction to classical statistics from a unified point of view. Different optimality criteria are considered, and methods for determining optimum procedures according to these criteria are developed. The application of the resulting theory to a variety of specific problems as an introduction to statistical methodology constitutes a second major theme.

On the other hand, the two books are essentially independent of each other. (As a result, there is some overlap in the preparatory chapters; also, each volume contains cross-references to related topics in the other.) They can therefore be taught in either order. However, TPE is somewhat more discursive and written at a slightly lower mathematical level and, for this reason, may offer the better starting point.

The material of the two volumes combined somewhat exceeds what can be comfortably covered in a year's course meeting 3 hours a week, thus providing the instructor with some choice of topics to be emphasized. A one-semester course covering both estimation and testing, can be obtained, for example, by deleting all large-sample considerations, all nonparametric material, the sections concerned with simultaneous estimation and testing, the minimax chapter of TSH, and some of the applications. Such a course might consist of the following sections: TPE: Chap. 2, § 1 and a few examples from §§ 2, 3; Chap. 3, §§1–3; Chap. 4, §§ 1–4. TSH: Chap. 3, §§ 1–3, 5, 7 (without proof of Theorem 6); Chap. 4, §§1–6; Chap. 5, §§ 1–7; Chap. 6, §§1–6; Chap. 7, §§1–3, 5–9, together with material from the preparatory chapters (TSH Chap. 1, 2; TPE Chap. 1) as it is needed.

Theory of Point Estimation

Contents

CHAPTER 1

Preparations

1. THE PROBLEM

Statistics is concerned with the collection of data and with their analysis and interpretation. We shall not consider the problem of data collection in this book but shall take the data as given and ask what they have to tell us. The answer depends not only on the data, on what is being observed, but also on background knowledge of the situation; the latter is formalized in the assumptions with which the analysis is entered. We shall distinguish between three principal lines of approach.

Data analysis. Here the data are analyzed on their own terms, essentially without extraneous assumptions. The principal aim is the organization and summarization of the data in ways that bring out their main features and clarify their underlying structure.

Classical inference and decision theory. The observations are now postulated to be the values taken on by random variables which are assumed to follow a joint probability distribution, P, belonging to some known class $\mathcal{P}$. Frequently, the distributions are indexed by a parameter, say θ (not necessarily real-valued), taking values in a set, Ω, so that

$$\mathcal{P} = \{P_\theta, \theta \in \Omega\}. \tag{1}$$

The aim of the analysis is then to specify a plausible value for θ (this is the problem of point estimation), or at least to determine a subset of Ω of which we can plausibly assert that it does, or does not, contain θ (estimation by confidence sets or hypothesis testing). Such a statement about θ can be viewed as a summary of the information provided by the data and may be used as a guide to action.

Bayesian analysis. In this approach, it is assumed in addition that θ is itself a random variable (though unobservable) with a *known* distribution. This prior distribution (specified prior to the availability of the data) is