SCHAUM'S OUTLINE SERIES

THEORY AND PROBLEMS OF

ELECTROMAGNETICS

JOSEPH A. EDMINISTER

INCLUDING 310 SOLVED PROBLEMS

SCHAUM'S OUTLINE SERIES IN ENGINEERING

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SCHAUM'S OUTLINE OF

THEORY AND PROBLEMS

of

ELECTROMAGNETICS

by

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Preface

This book is intended to serve as a supplement to any of the introductory textbooks in electromagnetic field theory for engineers; it may also be used by itself as the text for a brief first course. As in other Schaum's Outlines the emphasis is on how to solve problems. Each chapter consists of an ample set of problems with detailed solutions, and a further set of problems with answers, preceded by a simplified outline of the principles and facts needed to understand the problems and their solutions. Although electromagnetic problems of the physical world tend to be quite elaborate, it was decided in this book to present mostly short, single-concept problems. It is felt that this will prove advantageous to the student who seeks help on a particular point, as well as to those who may use the book for review purposes.

Throughout the book the mathematics has been kept as simple as possible, and an abstract approach has been avoided. Concrete examples are liberally used and numerous graphs and sketches are given. I have found in many years of teaching that the solution of most problems begins with a carefully drawn sketch.

This book is dedicated to my students, who have shown me where the difficulties in the subject lie. For editorial assistance I want to express my gratitude to the staff of McGraw-Hill. Sincere thanks to Thomas R. Connell for his great care in checking all the problems and offering suggestions. Eileen Kerns deserves thanks for her capable typing of the manuscript. And finally, thanks are due to my family, in particular my wife Nina, for constant support and encouragement, without which the book could not have been written.

JOSEPH A. EDMINISTER

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Chapter 1

Vector Analysis

1.1 VECTOR NOTATION

In order to distinguish *vectors* (quantities having magnitude and direction) from *scalars* (quantities having magnitude only) the vectors are denoted by boldface symbols. A *unit vector*, one of absolute value (or magnitude or length) 1, will in this book always be indicated by a boldface, lowercase **a**. The unit vector in the direction of a vector **A** is determined by dividing **A** by its absolute value:

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} \quad \text{or} \quad \frac{\mathbf{A}}{A}$$

where $|\mathbf{A}| = A = \sqrt{\mathbf{A}\cdot\mathbf{A}}$ (see Section 1.2).

By use of the unit vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ along the x, y, and z axes of a cartesian coordinate system, an arbitrary vector can be written in *component form*:

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

1.2 VECTOR ALGEBRA

1. Vectors may be added and subtracted.

$$\begin{aligned}\mathbf{A} \pm \mathbf{B} &= (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \pm (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z)\\ &= (A_x \pm B_x)\mathbf{a}_x + (A_y \pm B_y)\mathbf{a}_y + (A_z \pm B_z)\mathbf{a}_z\end{aligned}$$

2. The associative, distributive, and commutative laws apply.

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

$$k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B} \qquad (k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$$

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

3. The *dot product* of two vectors is, by definition,

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \qquad \text{(read "A dot B")}$$

where θ is the smaller angle between **A** and **B**. From the component form it can be shown that

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

In particular,

$$\mathbf{A}\cdot\mathbf{A} = |\mathbf{A}|^2 = A_x^2 + A_y^2 + A_z^2$$

4. The *cross product* of two vectors is, by definition,

$$\mathbf{A}\times\mathbf{B} = (AB\sin\theta)\mathbf{a}_n \qquad \text{(read "A cross B")}$$

where θ is the smaller angle between **A** and **B**, and $\mathbf{a}_n$ is a unit vector normal to the plane determined by **A** and **B** when they are drawn from a common point. There are two normals to the plane, so further specification is needed. The normal selected is the one in the direction of advance of a right-hand screw when **A** is turned toward **B** (Fig. 1-1). Because of this direction requirement, the commutative law does not apply to the cross product; instead,

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

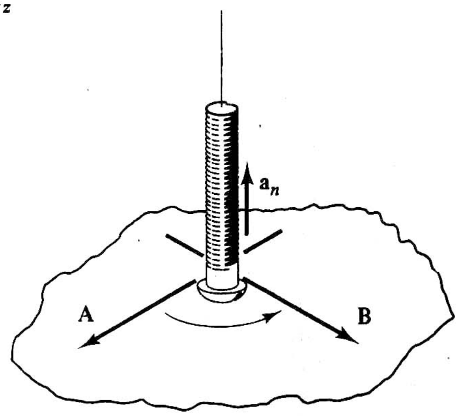


Fig. 1-1

Expanding the cross product in component form,

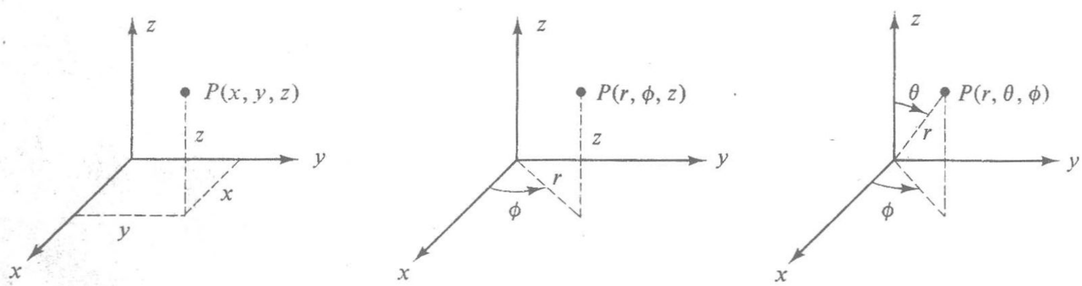
$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\times(B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z)\\ &= (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z\end{aligned}$$

which is conveniently expressed as a determinant:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

1.3 COORDINATE SYSTEMS

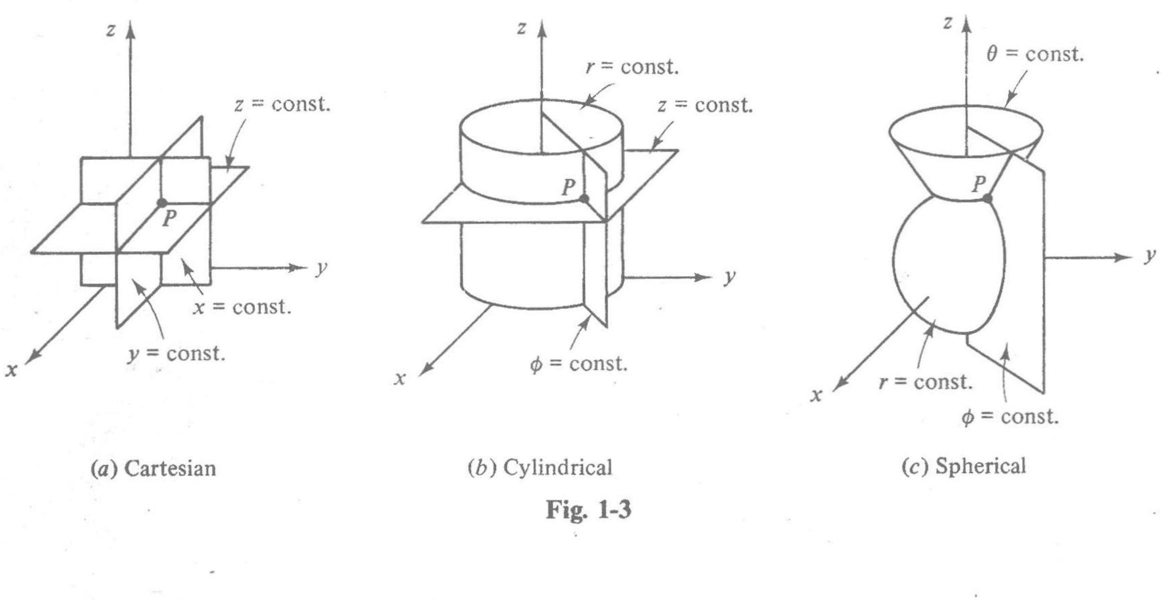
A problem which has cylindrical or spherical symmetry could be expressed and solved in the familiar cartesian coordinate system. However, the solution would fail to show the symmetry and in most cases would be needlessly complex. Therefore, throughout this book, in addition to the cartesian coordinate system, the circular cylindrical and the spherical coordinate systems will be used. All three will be examined together in order to illustrate the similarities and the differences.



(*a*) Cartesian (*b*) Cylindrical (*c*) Spherical

Fig. 1-2

A point P is described by three coordinates, in cartesian (x, y, z), in circular cylindrical (r, ϕ, z) and in spherical (r, θ, ϕ), as shown in Fig. 1-2. The order of specifying the coordinates is important and should be carefully followed. The angle ϕ is the same angle in both the cylindrical and spherical systems. But, in the order of the coordinates, ϕ appears in the second position in cylindrical, (r, ϕ, z), and the third position in spherical, (r, θ, ϕ). The same symbol, r, is used in both cylindrical and

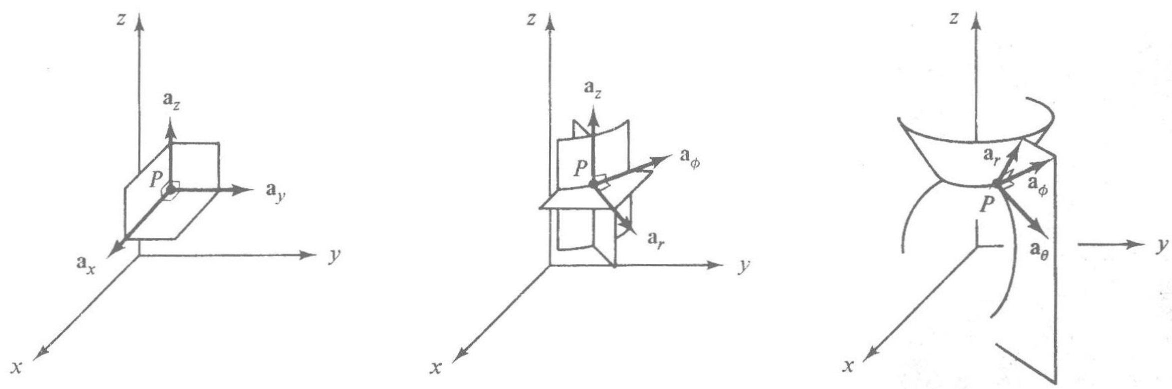


(*a*) Cartesian (*b*) Cylindrical (*c*) Spherical

Fig. 1-3

spherical for two quite different things. In cylindrical coordinates r measures the distance from the z axis in a plane normal to the z axis, while in the spherical system r measures the distance from the origin to the point. It should be clear from the context of the problem which r is intended.

A point is also defined by the intersection of three orthogonal surfaces, as shown in Fig. 1-3. In cartesian coordinates the surfaces are the infinite planes $x = \text{const.}$, $y = \text{const.}$, and $z = \text{const.}$ In cylindrical coordinates, $z = \text{const.}$ is the same infinite plane as in cartesian; $\phi = \text{const.}$ is a half plane with its edge along the z axis; $r = \text{const.}$ is a right circular cylinder. These three surfaces are orthogonal and their intersection locates point P. In spherical coordinates, $\phi = \text{const.}$ is the same half plane as in cylindrical; $r = \text{const.}$ is a sphere with its center at the origin; $\theta = \text{const.}$ is a right circular cone whose axis is the z axis and whose vertex is at the origin. Note that θ is limited to the range $0 \le \theta \le \pi$.



(*a*) Cartesian (*b*) Cylindrical (*c*) Spherical

Fig. 1-4

Figure 1-4 shows the three unit vectors at point P. In the cartesian system the unit vectors have fixed directions, independent of the location of P. This is not true for the other two systems (except in the case of $\mathbf{a}_z$). Each unit vector is normal to its coordinate surface and is in the direction in which the coordinate increases. Notice that all these systems are right-handed:

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \qquad \mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z \qquad \mathbf{a}_r \times \mathbf{a}_\theta = \mathbf{a}_\phi$$

The component forms of a vector in the three systems are

$$\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \qquad \text{(cartesian)}$$
$$\mathbf{A} = A_r \mathbf{a}_r + A_\phi \mathbf{a}_\phi + A_z \mathbf{a}_z \qquad \text{(cylindrical)}$$
$$\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi \qquad \text{(spherical)}$$

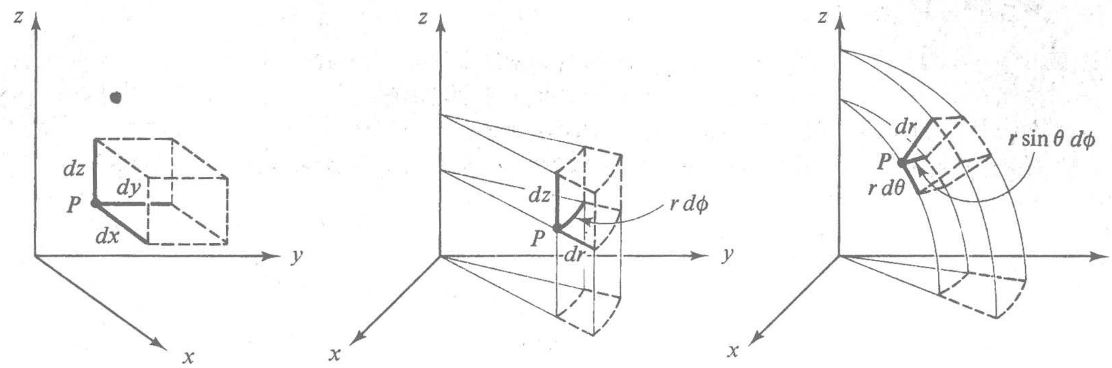
It should be noted that the components A_x, A_r, A_ϕ, etc., are not generally constants but more often are functions of the coordinates in that particular system.

1.4 DIFFERENTIAL VOLUME, SURFACE, AND LINE ELEMENTS

When the coordinates of point P are expanded to $(x + dx, y + dy, z + dz)$ or $(r + dr, \phi + d\phi, z + dz)$ or $(r + dr, \theta + d\theta, \phi + d\phi)$ a differential volume dv is formed. To the first order in infinitesimal quantities the differential volume is, in all three coordinate systems, a rectangular box. The value of dv in each system is given in Fig. 1-5.

From Fig. 1-5 may also be read the areas of the surface elements that bound the differential volume. For instance, in spherical coordinates, the differential surface element perpendicular to $\mathbf{a}_r$ is

$$dS = (r\,d\theta)(r \sin\theta\,d\phi) = r^2 \sin\theta\,d\theta\,d\phi$$



$dv \doteq dx\,dy\,dz$ (*a*) Cartesian

$dv = r\,dr\,d\phi\,dz$ (*b*) Cylindrical

$dv = r^2 \sin\theta\,dr\,d\theta\,d\phi$ (*c*) Spherical

Fig. 1-5

The differential line element, $d\ell$, is the diagonal through P. Thus

$$d\ell^2 = dx^2 + dy^2 + dz^2 \qquad \text{(cartesian)}$$
$$d\ell^2 = dr^2 + r^2\,d\phi^2 + dz^2 \qquad \text{(cylindrical)}$$
$$d\ell^2 = dr^2 + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2 \qquad \text{(spherical)}$$

1.5 VECTOR FIELDS

The vector expressions in electromagnetics are generally such that the coefficients of the unit vectors contain the variables. Therefore, the expression changes its magnitude and direction from point to point throughout the region of interest.

Consider, for example, the vector

$$\mathbf{E} = -x\mathbf{a}_x + y\mathbf{a}_y$$

Values of x and y may be substituted into the expression to give $\mathbf{E}$ at the various locations. After a number of points are examined, the pattern becomes evident. Figure 1-6 shows this field.

In addition, a vector field may vary with time. Thus, the two-dimensional field examined above could be given a time variation such as

$$\mathbf{E} = (-x\mathbf{a}_x + y\mathbf{a}_y)\sin\omega t$$

or

$$\mathbf{E} = (-x\mathbf{a}_x + y\mathbf{a}_y)e^{j\omega t}$$

The electric and magnetic fields of the later chapters are all time-variable. And, as might be expected, they will be differentiated with respect to time and also integrated with respect to time. However, both operations will follow naturally and seldom cause any great difficulty.

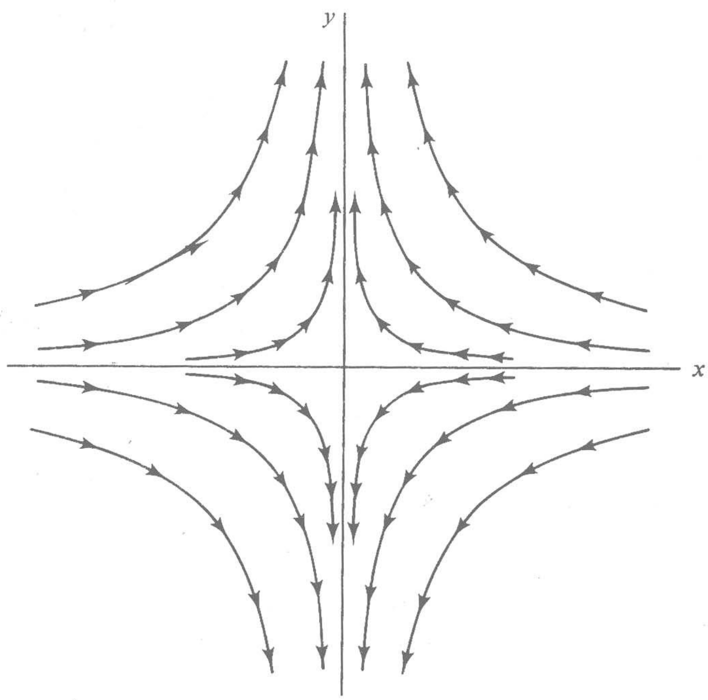


Fig. 1-6

1.6 TRANSFORMATIONS

The vector or vector field in a particular problem exists in the physical world and the coordinate system which is employed to express it is merely a frame of reference. A wise choice of the coordinate system at the outset will often result in a more direct solution to the problem and a concise final expression which shows the symmetry present. At times, however, it is necessary to transform a vector field in one system into another.

EXAMPLE 1 Consider

$$\mathbf{A} = 5r\mathbf{a}_r + 2\sin\phi\,\mathbf{a}_\theta + 2\cos\theta\,\mathbf{a}_\phi$$

in spherical coordinates. The variables r, θ, ϕ can be changed into cartesian by referring to Fig. 1-2 and applying basic trigonometry. Thus

$$r = \sqrt{x^2+y^2+z^2} \qquad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}} \qquad \tan\phi = \frac{y}{x}$$

Now the spherical components of the vector field $\mathbf{A}$ can be written in terms of x, y, and z:

$$\mathbf{A} = 5\sqrt{x^2+y^2+z^2}\,\mathbf{a}_r + \frac{2y}{\sqrt{x^2+y^2}}\mathbf{a}_\theta + \frac{2z}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_\phi$$

The unit vectors $\mathbf{a}_r$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ can also be transformed into their cartesian equivalents by referring to Fig. 1-4 and applying basic trigonometry. Thus

$$\mathbf{a}_r = \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_x + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_y + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_z$$

$$\mathbf{a}_\theta = \frac{xz}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\mathbf{a}_x + \frac{yz}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\mathbf{a}_y - \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\mathbf{a}_z$$

$$\mathbf{a}_\phi = \frac{-y}{\sqrt{x^2+y^2}}\mathbf{a}_x + \frac{x}{\sqrt{x^2+y^2}}\mathbf{a}_y$$

Combining these with the transformed components results in

$$\mathbf{A} = \left(5x + \frac{2xyz}{\sqrt{x^2+y^2+z^2}\,(x^2+y^2)} - \frac{2yz}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\right)\mathbf{a}_x$$
$$+ \left(5y + \frac{2y^2z}{(x^2+y^2)\sqrt{x^2+y^2+z^2}} + \frac{2xy}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}}\right)\mathbf{a}_y$$
$$+ \left(5z - \frac{2y}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{a}_z$$

Solved Problems

1.1 Show that the vector directed from $M(x_1, y_1, z_1)$ to $N(x_2, y_2, z_2)$ in Fig. 1-7 is given by

$$(x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

The coordinates of M and N are used to write the two position vectors $\mathbf{A}$ and $\mathbf{B}$ in Fig. 1-7.

$$\mathbf{A} = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z$$
$$\mathbf{B} = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$$

Then

$$\mathbf{B} - \mathbf{A} = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

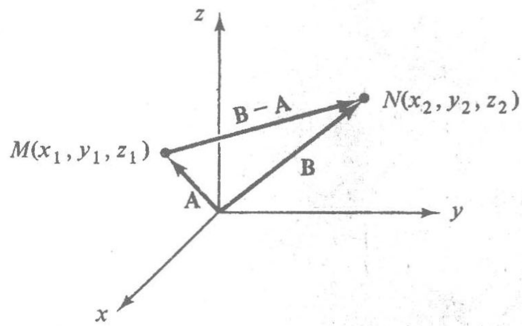


Fig. 1-7

1.2. Find the vector **A** directed from $(2, -4, 1)$ to $(0, -2, 0)$ in cartesian coordinates and find the unit vector along **A**.

$$\mathbf{A} = (0-2)\mathbf{a}_x + (-2-(-4))\mathbf{a}_y + (0-1)\mathbf{a}_z = -2\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$$

$$|\mathbf{A}|^2 = (-2)^2 + (2)^2 + (-1)^2 = 9$$

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = -\frac{2}{3}\mathbf{a}_x + \frac{2}{3}\mathbf{a}_y - \frac{1}{3}\mathbf{a}_z$$

1.3. Find the distance between $(5, 3\pi/2, 0)$ and $(5, \pi/2, 10)$ in cylindrical coordinates.

First, obtain the *cartesian* position vectors **A** and **B** (see Fig. 1-8).

$$\mathbf{A} = -5\mathbf{a}_y \qquad \mathbf{B} = 5\mathbf{a}_y + 10\mathbf{a}_z$$

Then $\mathbf{B} - \mathbf{A} = 10\mathbf{a}_y + 10\mathbf{a}_z$ and the required distance between the points is

$$|\mathbf{B} - \mathbf{A}| = 10\sqrt{2}$$

The cylindrical coordinates of the points cannot be used to obtain a vector between the points in the same manner as was employed in Problem 1.1 in cartesian coordinates.

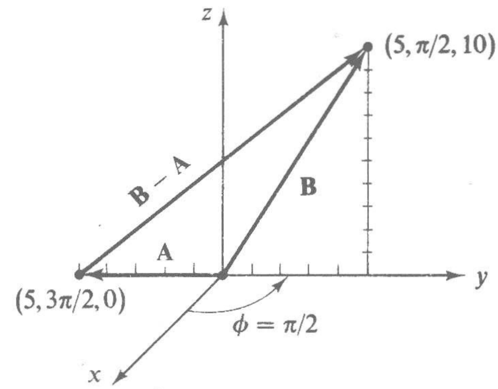


Fig. 1-8

1.4. Show that $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$.

Express the dot product in component form.

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z) \cdot (B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z)\\ &= (A_x\mathbf{a}_x) \cdot (B_x\mathbf{a}_x) + (A_x\mathbf{a}_x) \cdot (B_y\mathbf{a}_y) + (A_x\mathbf{a}_x) \cdot (B_z\mathbf{a}_z)\\ &\quad + (A_y\mathbf{a}_y) \cdot (B_x\mathbf{a}_x) + (A_y\mathbf{a}_y) \cdot (B_y\mathbf{a}_y) + (A_y\mathbf{a}_y) \cdot (B_z\mathbf{a}_z)\\ &\quad + (A_z\mathbf{a}_z) \cdot (B_x\mathbf{a}_x) + (A_z\mathbf{a}_z) \cdot (B_y\mathbf{a}_y) + (A_z\mathbf{a}_z) \cdot (B_z\mathbf{a}_z)\end{aligned}$$

However, $\mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1$ because the $\cos\theta$ in the dot product is unity when the angle is zero. And when $\theta = 90°$, $\cos\theta$ is zero. Hence all other dot products of the unit vectors are zero. Thus

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

1.5. Given $\mathbf{A} = 2\mathbf{a}_x + 4\mathbf{a}_y - 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - \mathbf{a}_y$, find $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$.

$$\mathbf{A} \cdot \mathbf{B} = (2)(1) + (4)(-1) + (-3)(0) = -2$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 4 & -3 \\ 1 & -1 & 0 \end{vmatrix} = -3\mathbf{a}_x - 3\mathbf{a}_y - 6\mathbf{a}_z$$

1.6. Show that $\mathbf{A} = 4\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z$ are perpendicular.

Since the dot product contains $\cos\theta$, a dot product of zero from any two nonzero vectors implies that $\theta = 90°$.

$$\mathbf{A} \cdot \mathbf{B} = (4)(1) + (-2)(4) + (-1)(-4) = 0$$

1.7. Given $\mathbf{A} = 2\mathbf{a}_x + 4\mathbf{a}_y$ and $\mathbf{B} = 6\mathbf{a}_y - 4\mathbf{a}_z$, find the smaller angle between them using (*a*) the cross product, (*b*) the dot product.

(*a*)

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 4 & 0 \\ 0 & 6 & -4 \end{vmatrix} = -16\mathbf{a}_x + 8\mathbf{a}_y + 12\mathbf{a}_z$$

$$|\mathbf{A}| = \sqrt{(2)^2 + (4)^2 + (0)^2} = 4.47$$

$$|\mathbf{B}| = \sqrt{(0)^2 + (6)^2 + (-4)^2} = 7.21$$

$$|\mathbf{A} \times \mathbf{B}| = \sqrt{(-16)^2 + (8)^2 + (12)^2} = 21.54$$

Then, since $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}| \sin\theta$,

$$\sin\theta = \frac{21.54}{(4.47)(7.21)} = 0.668 \qquad \text{or} \qquad \theta = 41.9°$$

(*b*)

$$\mathbf{A} \cdot \mathbf{B} = (2)(0) + (4)(6) + (0)(-4) = 24$$

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|\,|\mathbf{B}|} = \frac{24}{(4.47)(7.21)} = 0.745 \qquad \text{or} \qquad \theta = 41.9°$$

1.8. Given $\mathbf{F} = (y - 1)\mathbf{a}_x + 2x\mathbf{a}_y$, find the vector at $(2, 2, 1)$ and its projection on $\mathbf{B}$, where $\mathbf{B} = 5\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$.

$$\mathbf{F}(2, 2, 1) = (2 - 1)\mathbf{a}_x + (2)(2)\mathbf{a}_y$$
$$= \mathbf{a}_x + 4\mathbf{a}_y$$

As indicated in Fig. 1-9, the projection of one vector on a second vector is obtained by expressing the unit vector in the direction of the second vector and taking the dot product.

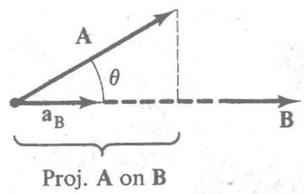


Fig. 1-9

$$\text{Proj. } \mathbf{A} \text{ on } \mathbf{B} = \mathbf{A} \cdot \mathbf{a}_B = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{B}|}$$

Thus, at $(2, 2, 1)$,

$$\text{Proj. } \mathbf{F} \text{ on } \mathbf{B} = \frac{\mathbf{F} \cdot \mathbf{B}}{|\mathbf{B}|} = \frac{(1)(5) + (4)(-1) + (0)(2)}{\sqrt{30}} = \frac{1}{\sqrt{30}}$$

1.9. Given $\mathbf{A} = \mathbf{a}_x + \mathbf{a}_y$, $\mathbf{B} = \mathbf{a}_x + 2\mathbf{a}_z$, and $\mathbf{C} = 2\mathbf{a}_y + \mathbf{a}_z$, find $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and compare it with $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

Then

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -2 & -1 \\ 0 & 2 & 1 \end{vmatrix} = -2\mathbf{a}_y + 4\mathbf{a}_z$$

A similar calculation gives $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = 2\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$. Thus the parentheses that indicate which cross product is to be taken first are essential in the vector triple product.

1.10. Using the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ of Problem 1.9, find $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ and compare it with $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$.

From Problem 1.9, $\mathbf{B} \times \mathbf{C} = -4\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z$. Then

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = (1)(-4) + (1)(-1) + (0)(2) = -5$$

Also from Problem 1.9, $\mathbf{A} \times \mathbf{B} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$. Then

$$\mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = (2)(0) + (-2)(2) + (-1)(1) = -5$$

Parentheses are not needed in the scalar triple product since it has meaning only when the cross product is taken first. In general, it can be shown that

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

As long as the vectors appear in the same cyclic order the result is the same. The scalar triple products out of this cyclic order have a change in sign.

1.11. Express the unit vector which points from $z = h$ on the z axis toward $(r, \phi, 0)$ in cylindrical coordinates. See Fig. 1-10.

The vector **R** is the difference of two vectors:

$$\mathbf{R} = r\mathbf{a}_r - h\mathbf{a}_z$$

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{r\mathbf{a}_r - h\mathbf{a}_z}{\sqrt{r^2 + h^2}}$$

The angle ϕ does not appear explicitly in these expressions. Nevertheless, both **R** and $\mathbf{a}_R$ vary with ϕ through $\mathbf{a}_r$.

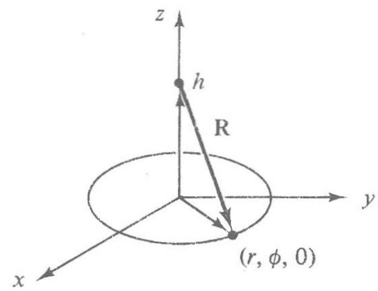


Fig. 1-10

1.12. Express the unit vector which is directed toward the origin from an arbitrary point on the plane $z = -5$, as shown in Fig. 1-11.

Since the problem is in cartesian coordinates, the two-point formula of Problem 1.1 applies.

$$\mathbf{R} = -x\mathbf{a}_x - y\mathbf{a}_y + 5\mathbf{a}_z$$

$$\mathbf{a}_R = \frac{-x\mathbf{a}_x - y\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{x^2 + y^2 + 25}}$$

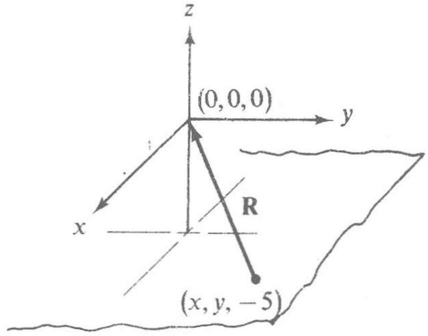


Fig. 1-11

1.13. Use the spherical coordinate system to find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius a (Fig. 1-12). What results when $\alpha = 0$ and $\beta = \pi$?

The differential surface element is [see Fig. 1-5(*c*)]

$$dS = r^2 \sin\theta \, d\theta \, d\phi$$

Then

$$A = \int_0^{2\pi} \int_\alpha^\beta a^2 \sin\theta \, d\theta \, d\phi$$
$$= 2\pi a^2(\cos\alpha - \cos\beta)$$

When $\alpha = 0$ and $\beta = \pi$, $A = 4\pi a^2$, the surface area of the entire sphere.

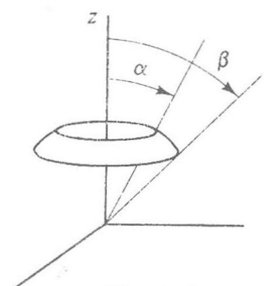


Fig. 1-12

1.14. Develop the equation for the volume of a sphere of radius a from the differential volume.

From Fig. 1-5(*c*), $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$. Then

$$v = \int_0^{2\pi} \int_0^{\pi} \int_0^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{4}{3}\pi a^3$$

1.15. Use the cylindrical coordinate system to find the area of the curved surface of a right circular cylinder where $r = 2$ m, $h = 5$ m, and $30° \leq \phi \leq 120°$ (see Fig. 1-13).

The differential surface element is $dS = r\,d\phi\,dz$. Then

$$A = \int_0^5 \int_{\pi/6}^{2\pi/3} 2\,d\phi\,dz = 5\pi \text{ m}^2$$

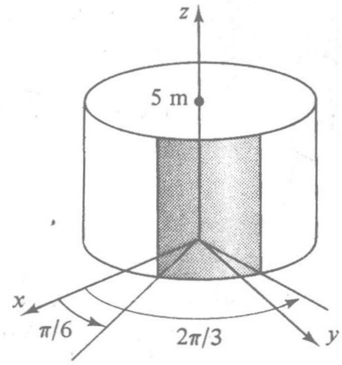


Fig. 1-13

1.16. Transform

$$\mathbf{A} = y\mathbf{a}_x + x\mathbf{a}_y + \frac{x^2}{\sqrt{x^2+y^2}}\mathbf{a}_z$$

from cartesian to cylindrical coordinates.

Referring to Fig. 1-2(*b*),

$$x = r\cos\phi \qquad y = r\sin\phi \qquad r = \sqrt{x^2+y^2}$$

Hence $$\mathbf{A} = r\sin\phi\,\mathbf{a}_x + r\cos\phi\,\mathbf{a}_y + r\cos^2\phi\,\mathbf{a}_z$$

Now the projections of the cartesian unit vectors on $\mathbf{a}_r$, $\mathbf{a}_\phi$, and $\mathbf{a}_z$ are obtained:

$$\begin{array}{lll} \mathbf{a}_x\cdot\mathbf{a}_r = \cos\phi & \mathbf{a}_x\cdot\mathbf{a}_\phi = -\sin\phi & \mathbf{a}_x\cdot\mathbf{a}_z = 0 \\ \mathbf{a}_y\cdot\mathbf{a}_r = \sin\phi & \mathbf{a}_y\cdot\mathbf{a}_\phi = \cos\phi & \mathbf{a}_y\cdot\mathbf{a}_z = 0 \\ \mathbf{a}_z\cdot\mathbf{a}_r = 0 & \mathbf{a}_z\cdot\mathbf{a}_\phi = 0 & \mathbf{a}_z\cdot\mathbf{a}_z = 1 \end{array}$$

Therefore

$$\mathbf{a}_x = \cos\phi\,\mathbf{a}_r - \sin\phi\,\mathbf{a}_\phi$$
$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_r + \cos\phi\,\mathbf{a}_\phi$$
$$\mathbf{a}_z = \mathbf{a}_z$$

and $$\mathbf{A} = 2r\sin\phi\cos\phi\,\mathbf{a}_r + (r\cos^2\phi - r\sin^2\phi)\mathbf{a}_\phi + r\cos^2\phi\,\mathbf{a}_z$$

1.17. A vector of magnitude 10 points from $(5, 5\pi/4, 0)$ in cylindrical coordinates toward the origin (Fig. 1-14). Express the vector in cartesian coordinates.

In cylindrical coordinates, the vector may be expressed as $10\mathbf{a}_r$, where $\phi = \pi/4$. Hence

$$A_x = 10\cos\frac{\pi}{4} = \frac{10}{\sqrt{2}} \qquad A_y = 10\sin\frac{\pi}{4} = \frac{10}{\sqrt{2}} \qquad A_z = 0$$

so that

$$\mathbf{A} = \frac{10}{\sqrt{2}}\mathbf{a}_x + \frac{10}{\sqrt{2}}\mathbf{a}_y$$

Notice that the value of the radial coordinate, 5, is immaterial.

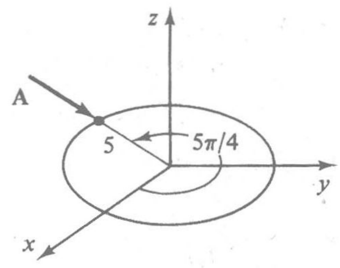


Fig. 1-14

Supplementary Problems

1.18. Given $\mathbf{A} = 4\mathbf{a}_y + 10\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + 3\mathbf{a}_y$, find the projection of **A** on **B**. *Ans.* $12/\sqrt{13}$

1.19. Given $\mathbf{A} = (10/\sqrt{2})(\mathbf{a}_x + \mathbf{a}_z)$ and $\mathbf{B} = 3(\mathbf{a}_y + \mathbf{a}_z)$, express the projection of **B** on **A** as a vector in the direction of **A**. *Ans.* $1.50\,(\mathbf{a}_x + \mathbf{a}_z)$

1.20. Find the angle between $\mathbf{A} = 10\mathbf{a}_y + 2\mathbf{a}_z$ and $\mathbf{B} = -4\mathbf{a}_y + 0.5\,\mathbf{a}_z$ using both the dot product and the cross product. *Ans.* 161.5°

1.21. Find the angle between $\mathbf{A} = 5.8\,\mathbf{a}_y + 1.55\,\mathbf{a}_z$ and $\mathbf{B} = -6.93\,\mathbf{a}_y + 4.0\,\mathbf{a}_z$ using both the dot product and the cross product. *Ans.* 135°

1.22. Given the plane $4x + 3y + 2z = 12$, find the unit vector normal to the surface in the direction away from the origin. *Ans.* $(4\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z)/\sqrt{29}$

1.23. Show that the vector fields **A** and **B** are everywhere perpendicular if $A_x B_x + A_y B_y + A_z B_z = 0$.

1.24. Find the relationship which the cartesian components of **A** and **B** must satisfy if the vector fields are everywhere parallel.

Ans. $\dfrac{A_x}{B_x} = \dfrac{A_y}{B_y} = \dfrac{A_z}{B_z}$

1.25. Express the unit vector directed toward the origin from an arbitrary point on the line described by $x = 0$, $y = 3$.

Ans. $\mathbf{a} = \dfrac{-3\mathbf{a}_y - z\mathbf{a}_z}{\sqrt{9 + z^2}}$

1.26. Express the unit vector directed toward the point (x_1, y_1, z_1) from an arbitrary point in the plane $y = -5$.

Ans. $\mathbf{a} = \dfrac{(x_1 - x)\mathbf{a}_x + (y_1 + 5)\mathbf{a}_y + (z_1 - z)\mathbf{a}_z}{\sqrt{(x_1 - x)^2 + (y_1 + 5)^2 + (z_1 - z)^2}}$

1.27. Express the unit vector directed toward the point $(0, 0, h)$ from an arbitrary point in the plane $z = -2$. Explain the result as h approaches -2.

Ans. $\mathbf{a} = \dfrac{-x\mathbf{a}_x - y\mathbf{a}_y + (h + 2)\mathbf{a}_z}{\sqrt{x^2 + y^2 + (h + 2)^2}}$

1.28. Given $\mathbf{A} = 5\mathbf{a}_x$ and $\mathbf{B} = 4\mathbf{a}_x + B_y\mathbf{a}_y$, find B_y such that the angle between **A** and **B** is 45°. If **B** also has a term $B_z\mathbf{a}_z$, what relationship must exist between B_y and B_z? *Ans.* $B_y = \pm 4$, $\sqrt{B_y^2 + B_z^2} = 4$

1.29. Show that the absolute value of $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ is the volume of the parallelepiped with edges **A**, **B**, and **C**. (*Hint*: First show that the base has area $|\mathbf{B} \times \mathbf{C}|$.)

1.30. Given $\mathbf{A} = 2\mathbf{a}_x - \mathbf{a}_z$, $\mathbf{B} = 3\mathbf{a}_x + \mathbf{a}_y$, and $\mathbf{C} = -2\mathbf{a}_x + 6\mathbf{a}_y - 4\mathbf{a}_z$, show that **C** is ⊥ to both **A** and **B**.

1.31. Given $\mathbf{A} = \mathbf{a}_x - \mathbf{a}_y$, $\mathbf{B} = 2\mathbf{a}_z$, and $\mathbf{C} = -\mathbf{a}_x + 3\mathbf{a}_y$, find $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$. Examine other variations of the scalar triple product. *Ans.* -4

1.32. Using the vectors of Problem 1.31 find $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$. *Ans.* $-8\mathbf{a}_z$

1.33. Find the unit vector directed from $(2, -5, -2)$ toward $(14, -5, 3)$.

Ans. $\mathbf{a} = \dfrac{12}{13}\mathbf{a}_x + \dfrac{5}{13}\mathbf{a}_z$

1.34. Show why the method of Problem 1.1 cannot be used in cylindrical coordinates for the points (r_1, ϕ_1, z_1) and (r_2, ϕ_2, z_2). Examine the same question for spherical coordinates.

1.35. Verify that the distance d between the two points of Problem 1.34 is given by

$$d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2$$

1.36. Find the vector directed from $(10, 3\pi/4, \pi/6)$ to $(5, \pi/4, \pi)$, where the points are given in spherical coordinates.
Ans. $-9.66\,\mathbf{a}_x - 3.54\,\mathbf{a}_y + 10.61\,\mathbf{a}_z$

1.37. Find the distance between $(2, \pi/6, 0)$ and $(1, \pi, 2)$, where the points are given in cylindrical coordinates.
Ans. 3.53

1.38. Find the distance between $(1, \pi/4, 0)$ and $(1, 3\pi/4, \pi)$, where the points are given in spherical coordinates.
Ans. 2.0

1.39. Use spherical coordinates and integrate to find the area of the region $0 \le \phi \le \alpha$ on the spherical shell of radius a. What is the result when $\alpha = 2\pi$? *Ans.* $2\alpha a^2$, $A = 4\pi a^2$

1.40. Use cylindrical coordinates to find the area of the curved surface of a right circular cylinder of radius a and height h. *Ans.* $2\pi ah$

1.41. Use cylindrical coordinates and integrate to obtain the volume of the right circular cylinder of Problem 1.40.
Ans. $\pi a^2 h$

1.42. Use spherical coordinates to write the differential surface areas dS_1 and dS_2 and then integrate to obtain the areas of the surfaces marked *1* and *2* in Fig. 1-15.
Ans. $\pi/4$, $\pi/6$

1.43. Use spherical coordinates to find the volume of a hemispherical shell of inner radius 2.00 m and outer radius 2.02 m. *Ans.* $0.162\pi\ \text{m}^3$

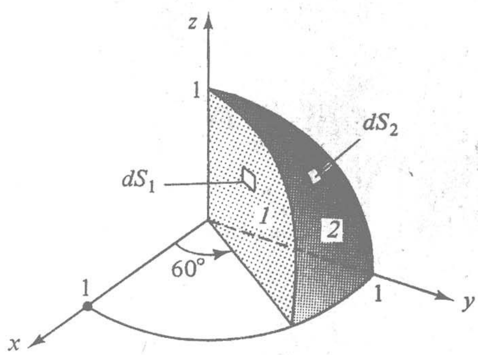


Fig. 1-15

1.44. Using spherical coordinates to express the differential volume, integrate to obtain the volume defined by $1 \le r \le 2$ m, $0 \le \theta \le \pi/2$, and $0 \le \phi \le \pi/2$. *Ans.* $\frac{7\pi}{6}\ \text{m}^3$

1.45. Transform the vector $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ into cylindrical coordinates.
Ans. $\mathbf{A} = (A_x\cos\phi + A_y\sin\phi)\mathbf{a}_r + (-A_x\sin\phi + A_y\cos\phi)\mathbf{a}_\phi + A_z\mathbf{a}_z$

1.46. Transform the vector $\mathbf{A} = A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$ into cartesian coordinates.

Ans.
$$\mathbf{A} = \left(\frac{A_r x}{\sqrt{x^2+y^2+z^2}} + \frac{A_\theta xz}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} - \frac{A_\phi y}{\sqrt{x^2+y^2}}\right)\mathbf{a}_x + \left(\frac{A_r y}{\sqrt{x^2+y^2+z^2}} + \frac{A_\theta yz}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} + \frac{A_\phi x}{\sqrt{x^2+y^2}}\right)\mathbf{a}_y + \left(\frac{A_r z}{\sqrt{x^2+y^2+z^2}} - \frac{A_\theta\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\right)\mathbf{a}_z$$