ADVANCES IN RESEARCH AND APPLICATIONS

Methods in COMPUTATIONAL PHYSICS

Edited by Bruce A. Bolt,
Berni Alder,
Sidney Fernbach,
and Manuel Rotenberg

VOLUME 11

Seismology: Surface Waves and Earth Oscillations

METHODS IN COMPUTATIONAL PHYSICS

Advances in Research and Applications

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Preface

Because a great many of the recent advances in seismology have depended on the high speed computer, it is an ideal topic for the serial publication *Methods in Computational Physics*. It is possible to select topics which define key seismological problems, and, at the same time, illustrate the numerical techniques found valuable in seismology. Hence Volumes 11 and 12, which together attempt to bring the main developments of seismology up to date, should prove themselves a useful text for seismologists, earthquake engineers, and graduate students in those subjects.

The five articles in Volume 11 deal with the computational analysis of surface waves and the eigenvibrations of the Earth. These subjects are related both historically and theoretically as is made clear in the review by Takeuchi and Saito. There are two principal types of seismic surface waves, called Love and Rayleigh waves. The corresponding free vibrations of the whole globe are of the torsional and spheriodal type. Progress in the calculation of Rayleigh waves from the original derivation in 1885 for a homogeneous elastic half-space had been such that textbooks in 1957 could only report the numerical results for the propagation through two plane parallel layers. By 1960, it had been shown that surface wave dispersion for an arbitrary number of parallel layers could be treated readily by loop repetition on a computer using a layer (or "transfer") matrix formation.

A further advance in the numerical modeling of geological structures is described in the article by D. M. Boore, where the appropriate partial differential equations with boundary conditions for heterogeneous materials are solved using a rather intricate finite difference scheme. A competing method, much used in structural engineering and soil mechanics, is described in the following paper by J. Lysmer and L. A. Drake. Their numerical procedure applies to linear viscoelastic Earth structures of rather general type. The irregular structure is replaced by a system of connected finite elements.

The computer techniques of processing seismograms to obtain information on the dispersion of seismic surface waves are presented by Dziewonski and Hales. Cross correlation is shown to be a basic tool. The significance to seismology of the Fast Fourier Transform (factorization method) is critically explored and examples are given of data processing at different frequencies. Fast algorithms for computation of eigenvalues is surface wave and terrestrial eigenvibration problems are explained by Schwab and Knopoff.

Some of the numerical methods discussed in this volume have not yet seen their full development. The thrust of future research will be to use seismological measurements to infer the physical properties of more realistic and refined three-dimensional Earth models. Lateral variation in upper mantle structure, oceanic–continental boundaries, plate boundaries, and mountain roots will be studied quantitatively for the first time using surface-wave dispersion. The effect of soils and local geological structure on strong earthquake shaking will be predicted by numerical methods.

Contents

SEISMIC SURFACE WAVES

H. Takeuchi and M. Saito

Finite Difference Methods for Seismic Wave Propagation in Heterogeneous Materials

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I. Introduction

AS MOST OF THE chapters in this book indicate, seismologists usually model the velocity and density structure of the earth with heterogeneity in the vertical direction only. Mathematical solutions for wave propagation in such models are relatively straightforward. There are a number of important problems in seismology, however, for which lateral changes in material properties are significant. Because the geometry in these cases cannot be represented as normal surfaces in a system of separable coordinates the solution of the direct problem is not simple, and some type of perturbation or numerical solution is necessary.

A number of different numerical schemes of varying complexity have been used to solve elastic wave propagation problems. In a continuing series of papers Alterman and her co-workers have used a simple finite difference method to solve the vector elastic equations of motion when subject to some specific initial and boundary conditions (Alterman and Karal, 1968; Alterman and Aboudi, 1969; Alterman *et al.*, 1972; see chapter by Alterman and Loewenthal in Volume 12 for a more complete list of Alterman's papers). This method was also used by Bertholf (1967) to solve for the transient displacements in an elastic finite cylindrical bar subjected to applied stresses at one end. Plamondon (1966) used a different, more complex method to compute the motion due to a spherical source beneath the earth's surface. Even more involved methods, which are capable of following the motion through regions of plastic, shock, or brittle behavior, have been devised by Maenchen and Sack (1963) and Petschek and Hansen (1968), among others. Another computational scheme which has been very successful in studying eigenvibration problems and is currently attracting much attention in seismology is the finite element method (see chapter by Lysmer and Drake, this volume).

Two other recently developed methods (not discussed in this volume) for wave propagation in laterally heterogeneous media are the wave scattering method of Aki and Larner (1970; Larner, 1970) and the perturbation method of Claerbout (1970a, b, 1971), Claerbout and Johnson (1972), and Landers (1971). These methods, potentially very valuable, have received little attention up to this time.

The usefulness of any of the above schemes depends greatly on the problem being solved; one must choose that method which gives reasonable answers with the least amount of storage space and computer time. The straightforward finite difference method discussed in this chapter is a practical way of solving a number of pertinent seismological problems. The essence of this technique is to replace the differential equations and boundary conditions by simple finite difference approximations in such a way that an explicit, recursive set of equations is formed. This results in a time-marching procedure which can be used to solve for the displacements at each grid point as a function of time given the motion at the first two time steps.

There are many advantages to the finite difference method discussed in this article. Some of these are that it is very easy to program, many different problems can be solved with only minor alterations of the program, and the preparation of input data for a particular problem is not tedious. Furthermore, as opposed to steady-state solutions, the use of transient signals gives information at many frequencies from one computer run. The transient signal, in combination with the explicit set of equations, also makes the treatment of artificial boundaries (required by computer storage space limitations) more natural and less worrisome than in most other numerical schemes. Another convenient feature is that displacements as a function of time at a given site or pictures of the total wave field at a given time can be obtained with equal ease.

The finite difference method will probably find its greatest use in solving problems not possessing analytical solutions, but it can also compete with the analytical solutions, especially when such solutions require the evaluation of complicated series expansions (Alterman and Karal, 1968). The ease with which it can be programmed makes the finite difference method an excellent pedagogical tool in illustrating concepts of wave propagation in a dynamic, controllable manner. It is particularly useful for this if propagation is restricted to one dimension only, for then the computations are very rapid.

The technique is limited, for practical reasons, to certain classes of problems. It is difficult to enumerate these here, but in a general way we can say that it is most useful in the near field region of sources, where the sources can be either real or, as in this chapter, effective sources introduced by complexities along the travel path. Thus, for example, it would be impractical to use the finite difference method to evaluate the surface displacements of a short period body wave incident upon an irregular crust–mantle interface. On the other hand, it is ideal for the solution of a layered model in which the layer thicknesses are on the order of the seismic wavelengths.

Finite difference methods for problems involving partial differential equations have been developed and used for years in such disciplines as meteorology and civil and mechanical engineering. To be useful in seismological problems, however, wave propagation in models having material property variations in at least two spatial dimensions must be treated. This requires large amounts of computer space and rapid calculations, and it was only several years ago that machines capable of handling such problems were commonly available (one of the first papers dealing with numerical wave propagation to appear in the seismological literature was by Cherry and Hurdlow in 1966). Although there is no lack of possible methods based on finite differences, relatively few have actually been tested and applied to nontrivial seismological problems. It is the goal of this chapter to present in detail the methods used and experience gained by the author in making several of these applications, with the hope that it will stimulate others to explore further the uses of the method. Several improvements included here have not been discussed by the author in previous publications. Theoretical aspects of finite difference solutions to partial differential equations have been avoided. For these, reference should be made to one of the textbooks on the subject (e.g., Richtmeyer and Morton, 1967).

II. Method

A. Assumptions

The basic problem concerns transient wave propagation in a semi-infinite half-space bounded by a stress free surface. The free surface need not be planar nor must the material making up the half-space be homogeneous. The material through which the waves propagate is assumed to be isotropic and linearly elastic (the treatment of viscoelastic material is discussed briefly in Section III,A). Because of storage space and computation time limitations we assume that all variations in material properties, boundaries, and wavefields take place in only two spatial directions (x, z).

With these assumptions the general elastic motion can be uncoupled into two types: horizontal shear motion (SH), characterized by displacements v in the y direction only, and coupled compressional and shear motion involving the x, z components of displacements u, w. Although many of the methods discussed below can be applied to the complete vector equation, this chapter will be concerned exclusively with SH motion. One of the primary reasons for this is that less storage space and computer time are required than in the corresponding vector elastic case, and thus more realistic heterogeneities can be modeled within the space–time limits available. Furthermore, the seismic radiation from earthquakes usually contains a significant amount of SH motion and it is SH motion that is of greatest interest in engineering seismology.

B. Derivative Approximations

1. *Standard Formulas*

The basis of the finite difference technique is the replacement of differential operators by difference approximations. These approximations can be found in a number of ways; here we only intend to introduce notation and present some formulas. Further details may be found in textbooks such as Smith (1965) and Mitchell (1969).

The continuous x, z, t space is divided into rectangular blocks. The displacement field is then specified by values at the discrete set of nodepoints represented by the corner intersections of the blocks. For constant x, z, and t spacing Δx, Δz, and Δt, any node is uniquely determined with reference to an arbitrary coordinate origin by the indices m, n, p. Thus $v_{m,n}^{p} = v(m\,\Delta x, n\,\Delta z, p\,\Delta t)$, where subscripts refer to spatial location and superscripts to time. The absence of an index implies that the variable represented by that index can take continuous values, as in $v_{m,n} = v(m\,\Delta x, n\,\Delta z, t)$. When interface conditions are discussed at a boundary between two media, the subscripts 1 and 2 will sometimes be used to denote the respective media. No confusion should exist with the more usual subscripts representing spatial location. As a final piece of nomenclature, in future discussions the term "computational star" will be used; this refers to the spatial pattern of gridpoints used in the difference approximation of a differential operator.

With the above notation, standard centered approximations for first and second derivatives are

$$(\partial v/\partial x)_m \simeq (v_{m+1/2} - v_{m-1/2})/\Delta x, \tag{1}$$

$$(\partial^2 v/\partial x^2)_m \simeq (v_{m+1} - 2v_m + v_{m-1})/(\Delta x)^2. \tag{2}$$

Another centered approximation to the first derivative is

$$(\partial v/\partial x)_m \simeq (v_{m+1} - v_{m-1})/2\Delta x. \tag{3}$$

We will also use single-sided approximations, such as

$$(\partial v/\partial x)_m \simeq (v_{m+1} - v_m)/\Delta x, \tag{4}$$

to the first derivative. These are of a lower order of accuracy than the centered approximations in Eqs. (1) and (3).

All of the above formulas apply, with obvious changes, to derivatives with respect to z and t. Formulas for nonconstant Δx, Δz, and Δt can also be found easily (e.g., Boore, 1970b; Rowe, 1955). For example, the formula for the second x-derivative is

$$\left(\frac{\partial^2 v}{\partial x^2}\right)_m \simeq 2\left[\frac{v_{m+1}}{h_2(h_1 + h_2)} - \frac{v_m}{h_1 h_2} + \frac{v_{m-1}}{h_1(h_1 + h_2)}\right], \tag{5}$$

where h_1, h_2 are the spacings between nodes $m - 1$, m and m, $m + 1$.

2. *Attempted Use of Splines*

For the wave equation in a homogeneous material we seek an approximation to the Laplacian operator acting on the displacement field at a given time. This is obtained by using Eq. (2) and a corresponding expression for the second z-derivative. Since one way of obtaining Eq. (2) is to differentiate an interpolating quadratic polynomial fit to the three points $m - 1$, m, $m + 1$, one might wonder if a better interpolating polynomial could be found which would give similar accuracy but with larger grid spacings. In this way a given spatial area could be represented by a smaller number of grid points and thus the computation time, which is proportional to the number of grid points, would be reduced. As an exploratory attempt, bicubic spline functions (Bhattacharyya, 1969) were fit to sets of points obtained by digitizing three cycles of a sine wave at different rates. Derivatives of the resulting spline function, evaluated at the node points, and difference approximations using Eq. (3) and Eq. (2) on the tabulated set of points from which the spline was generated, were then compared with the exact values. Figure 1 shows a measure of the mean percentage error, averaged over the second cycle (in order to avoid end condition effects) of the sine wave, as a function of digitized

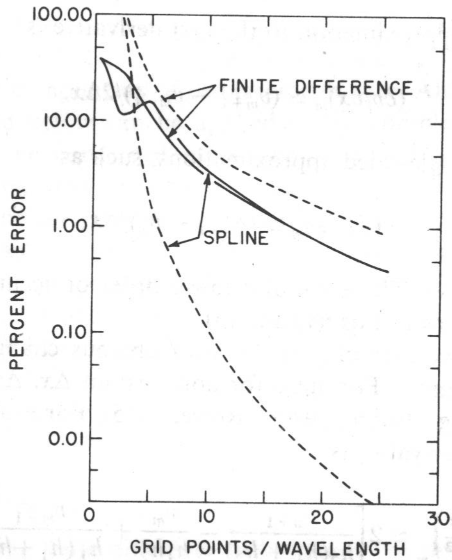

FIG. 1. The mean error, as a function of sampling rate, from spline and finite difference approximations to the first (- - -) and second (—) derivatives of a sine function.

points per wavelength. The spline gives a better approximation to the first derivative, but surprisingly, both the spline and finite difference approximation of the second derivative are nearly equivalent. Thus no obvious advantage would seem to accrue from splines as used here, especially considering that a matrix inversion (albeit a rapid one) is needed to generate the spline function. Splines, however, are finding utility in other areas of seismology, such as in the smoothing of travel time tables (Curtis and Shimshoni, 1970), calculation of divergence factors (Shimshoni and Ben-Menahem, 1970), calculation of ray theory amplitudes (Moler and Solomon, 1970), and location of small earthquakes (Wesson, 1971). The negative result obtained here does not imply that splines are not useful in other ways in the numerical solution of differential equations; books such as Schoenberg (1969) contain references to such techniques.

Disregarding splines, Fig. 1 shows the effect of grid spacing on the accuracy of the finite difference approximations (2) and (3). For example, at least 7 points are required per wavelength in order to obtain an accuracy of 95% in the second derivative. More discussion about the wavelength–gridspacing relationship will be found in Section II,E,1.

C. Equations of Motion

The basic equation for the displacement v in an inhomogeneous medium is

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial v}{\partial z}\right), \tag{6}$$

where $\mu(x, z)$ is the rigidity of the material and $\rho(x, z)$ is the density. Body forces (f) have been neglected; if present, an additional term ρf would be added to the right-hand side. Since wave propagation through homogeneous material joined along discrete interfaces is of most interest, a discussion of the general heterogeneous equation of motion will be deferred until the next section (which deals with boundary conditions). In a homogeneous material Eq. (6) becomes

$$\rho\, \partial^2 v/\partial t^2 = \mu \nabla^2 v. \tag{7}$$

where ∇^2 is the Laplacian operator. Replacing the derivatives by the difference approximation in Eq. (2), and gathering all the terms at time levels $p, p-1$ on the right-hand side, gives, as an approximation to Eq. (7),

$$v_{m,n}^{p+1} = 2v_{m,n}^{p} - v_{m,n}^{p-1} + \beta^2 \Delta t^2 \times \left[\frac{v_{m+1,n}^{p} - 2v_{m,n}^{p} + v_{m-1,n}^{p}}{(\Delta x)^2} + \frac{v_{m,n+1}^{p} - 2v_{m,n}^{p} + v_{m,n-1}^{p}}{(\Delta z)^2}\right], \tag{8}$$

where $\beta = (\mu/\rho)^{1/2}$ is the shear wave velocity. This is the basic equation used in the computations. It is explicit in the displacement at the new time level $p + 1$, and it is recursive; given initial displacements at two consecutive time points it is a simple matter to compute displacements at any other time by a forward time-marching process.

In common with most explicit finite difference approximations to partial differential equations, a condition relating the time and space grid intervals must be satisfied if the solution to the difference equations is to be stable. For the wave equation, this condition in practice is not excessively restrictive. This is in contrast to the heat flow equation, where the stability condition is so restrictive that implicit, unconditionally stable methods such as the alternating direction implicit scheme (Mitchell, 1969) must be used.

Various implicit methods, based on splitting the two-dimensional problem into several problems implicit in one direction only, do exist for the wave equation; Mitchell (1969) gives a thorough discussion of these schemes. Some are unconditionally stable and others, although requiring stability relations, are highly accurate. These schemes are all implicit and require a number of tridiagonal matrix inversions, for which there are very rapid algorithms, to progress from one time step to the next. Although more complicated than the explicit scheme given in Eq. (8), these methods may be useful in certain classes of problems. Because these schemes are in large part untested, however, there is a need for experimentation to determine their usefulness and limitations.

D. Boundary Conditions

1. *Physical Boundaries*

Although Eq. (6) in combination with initial conditions completely defines the problem, a special case arises where a discrete change in rigidity occurs across some surface in the body. Then Eq. (6) implies both

$$(\mu\, \partial v/\partial n)_{+} = (\mu\, \partial v/\partial n)_{-}, \tag{9}$$

where $\partial/\partial n$ is a derivative normal to the interface, and $v_{+} = v_{-}$. These conditions can also be expressed as the continuity of normal stress and displacement across the interface.

The explicit boundary condition at the stress free surface is

$$(\partial v/\partial n)_{\text{surf.}} = 0. \tag{10}$$

We can get this from Eq. (9) by assuming $(\mu)_{-} = 0$.

Most published applications of the finite difference method to elastic wave propagation involve plane, rather than curved, interfaces. For these, a number of methods which involve explicit approximation to the interface boundary condition (9) can be devised (Alterman and Karal, 1968; Bertholf, 1967; Boore, 1970a; Chiu, 1965). These approximations, however, are difficult to generalize to curved interfaces, and for this reason a relatively crude but adequate method was derived by the author. For want of a better name, this was called the explicit continuous stress method. Recently, several methods based on the heterogeneous wave equation have been investigated, and these appear to be superior in every respect to the explicit continuous stress method. Both of these approaches to curved boundaries are discussed below. Since these methods also work for plane interfaces (which are just a particular form of a curved boundary), the more specialized plane interface methods mentioned above will not be discussed.

a. *Heterogeneous Media Approach.* The interface condition (9) can be derived by considering the behavior of the equation describing the motion of a heterogeneous material, Eq. (6), as the distance over which the rigidity change occurs decreases to zero. This suggests that a natural way of treating the interface is to write approximations to Eq. (6) at the grid points near the interface. Two approximations are given below, and both reduce to Eq. (8) when the medium has uniform properties.

We are concerned only with the approximation of the right-hand side of Eq. (6); the time derivative can be replaced by the standard centered difference approximation. If the first derivative operator (1) is applied consecutively, the x-derivative is given by

$$\frac{\partial}{\partial x}\left(\mu \frac{\partial v}{\partial x}\right) \simeq \frac{\mu_{m+1/2}\, v_{m+1} - (\mu_{m+1/2} + \mu_{m-1/2}) v_m + \mu_{m-1/2}\, v_{m-1}}{(\Delta x)^2}. \quad (11)$$

The approximation of the z-derivative is similar. Since we have detailed knowledge of the rigidity for any point in space, evaluating it midway between grid points, as implied by $\mu_{m+1/2}$ and $\mu_{m-1/2}$, is not a problem.

Another approach which depends more on the detailed variation of $\mu(x, z)$ is due to Tikhonov and Samarskii (Mitchell, 1969, p. 23). To start, a variable w, defined by

$$w = -\mu(\partial v/\partial x), \quad (12)$$

is introduced. The equation above is rewritten

$$w/\mu = -\partial v/\partial x \quad (13)$$

and integrated over the interval $[(m-1)\,\Delta x, m\,\Delta x]$. Replacing w by a constant "mean-value" $w_{m-1/2}$ gives

$$w_{m-1/2} \int_{x_{m-1}}^{x_m} \frac{dx}{\mu(x, z)} = -(v_m - v_{m-1}), \quad (14)$$