

# STRATEGIC FINANCIAL PLANNING OVER THE LIFECYCLE

A Conceptual Approach to  
Personal Risk Management



Narat Charupat • Huaxiong Huang  
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# Strategic Financial Planning over the Lifecycle

*A Conceptual Approach to Personal Risk Management*

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## STRATEGIC FINANCIAL PLANNING OVER THE LIFECYCLE

This book on personal financial planning and wealth management employs the lifecycle model of financial economics. The central idea of “consumption smoothing” is used to connect chapters and topics such as saving and investment, debt management, risk management, and retirement planning. The first part of the book is nontechnical and aimed at a wide audience with no special technical background. The second part of the book provides a rigorous presentation of the lifecycle model from first principles using the calculus of variations. The accompanying website is found at [http://www.yorku.ca/milevsky/?page\\_id=185](http://www.yorku.ca/milevsky/?page_id=185).

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*To our wives...*

*. . . As far as I am aware, no one has challenged the view that if people were capable of it, they ought to plan their consumption, saving, and retirement according to the principles enunciated by Modigliani and Brumberg in the 1950s . . .*

**Angus S. Deaton**, Princeton University, 2005

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# ONE

## Introduction and Motivation

This book grew out of the deep frustration two of the three authors have experienced for many years, trying to teach a practical yet rigorous course on personal – in contrast to corporate or investment – finance, to undergraduate and graduate students at Canadian business schools. Although there are many college-level textbooks that discuss the *tactical* aspects of personal finance – nuggets such as: credit card debt is bad; reduce fees on mutual funds; regular saving is important; have a budget; and so on – we have not come across a textbook that integrated all these disparate concepts into a conceptual or *strategic* framework for financial decision making, *based on sound economic principles*.

For the most part, personal finance is being taught as a collection of stand-alone facts about “smart” money management. Most existing textbooks are written assuming a (very) basic background in mathematics on the part of the student, which limits the financial and economic level at which such a course can be delivered and the material discussed. Moreover, in today’s Google and YouTube world, curious students could obtain more relevant, accurate, and up-to-date information about most (if not all) of the products that are part of the personal financial toolkit. In our opinion, a textbook that allocates most of its pages to the *tactical* aspects of financial planning, such as explaining how to read a credit card statement or how to get a copy of your credit report from your local credit bureau, or how to open up a brokerage account, is not advanced enough for a third or fourth year course in a business school. Many personal finance textbooks – whose first editions were written before Internet browsers existed – are a relic from a bygone era in which good information could not be located on the Web within seconds.

On a related note – to make things more difficult for us as instructors – the recent “discoveries” by behavioral economists that individuals do not

adhere to the most basic axioms of rational choice, and the overwhelming evidence that most consumers make systematic and persistent mistakes with their money, create an even greater need for *a conceptual framework* that goes beyond amusing anecdotes regarding dollar bills left lying on the street.

Likewise, from the perspective of students in the engineering and sciences, it is often difficult for those who are interested in personal finance problems to find a background textbook that provides an overview of the relevant institutional features of personal finance and insurance together with the mathematical treatments used to solve these problems. Often, mathematics students have to sit through various courses offered in business schools, occasionally extracting useful information from a mountain of tangential material.

So, like any authors embarking on an ambitious writing project, we believe there is a niche to be filled. In particular, our goal in writing this book – and the way we approach the topic in class – is to *teach personal finance from the perspective of the lifecycle model* (LCM), originally formulated mathematically by Ramsey (1928), economically by Fisher (1930), then refined by Modigliani and Brumberg (1954) as well as Friedman (1957), and finally adjusted for lifetime uncertainty by Yaari (1965). Our intention is to extract as many practical insights as possible in an accessible and analytically tractable manner. If there is one question that links every single dilemma in personal finance, it is: **What course of action will help me maximize my standard of living – in the smoothest way possible – over the rest of my life?**

To avoid distraction, mathematical techniques are only presented when they are absolutely needed. Our emphasis is on the practical aspects of these techniques rather than mathematical rigor. The first twelve chapters of this book – which are geared toward undergraduates in business and finance – present the lifecycle model of investment and consumption under very simple assumptions about wages, retirement dates, and investment returns. The final two chapters (13 and 14) are (much) more mathematical and present advanced material related to the LCM, leading up to the Merton (1990) work on asset allocation in continuous time. The two final chapters are more suitable for an advanced undergraduate audience in economics and applied mathematics, or perhaps a first year graduate course, assuming they have the mathematical maturity and interest. As far as the numerical examples and case studies are concerned, we focus our examples on the Canadian environment (and in particular the tax material in Chapter 6) mainly because this is where we are located and where we currently teach. The other chapters or sections that contain a substantial amount of Canadian

content are so designated. That said, most of the conceptual material – which forms the majority of this book – is universal enough to apply anywhere.

In writing this book we aimed to “prove” that personal finance can be taught to university students in an intellectually satisfying manner, within a rational and strategic framework. We hope you agree.



## TWO

### Mathematical Preliminaries – Working with Interest Rates

#### Learning Objectives

In this chapter, we will review the concepts of interest rates and time value of money (TVM). There are several types of present-value and future-value formulas, each of which is used in specific circumstances. Our goal is to make sure that you understand when (i.e., in what context) these formulas should be used. A good understanding of this chapter is needed to proceed to future chapters, where we will need to calculate the amounts of your consumption and savings at various points in time.

Although we believe that most of you have covered these materials in your previous finance courses, we recommend that you take another look at them and familiarize yourself with the notations we will use in the rest of this book.

#### 2.1 Interest Rates

As you may recall, an interest rate is the rate of return that a borrower promises to pay for the use of money that he or she borrows from the lender. Normally, it is expressed in terms of per-annum percentage rates (e.g., 4% p.a.). To express it properly, however, we also need to state the compounding frequency of the rate, which is the number of compounding periods in one year. In other words, it is the number of times in a year that interest is calculated and added to the principal of the loan.

For example, annual compounding means that interest is added to the principal once a year. Suppose you invest \$1 for one year at the interest rate of 4% p.a., annual compounding. At the end of the year, you will receive:

$$(1 + 0.04)^1 = 1.04.$$