ENERGY METHODS IN STRESS ANALYSIS

T. H. RICHARDS

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with an introduction to finite element techniques

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Preface

Mechanics has developed as a branch of rational thought from very early times; it is the oldest of the physical sciences and its principles, formulated to describe mechanical behaviour in nature, form the basis of engineering calculations for a vast range of devices and structures. In applying these principles to mechanical design, an essential activity is the formulation of conceptual models, or idealizations, of real situations to which one's current stock of analytic techniques may be applied: the greater the precision one seeks in the correlation between actual and predicted behaviour, the more sophisticated does the model need to be. Unfortunately, this increased sophistication almost always incurs increased complexity in analysis and it is this which ultimately places a practical limit on the implementation of advanced methods to design. However, the developments which have taken place in the capability of computing devices in recent years have made possible the implementation of advanced methods yielding levels of precision which previously could not have been contemplated. From the point of view both of formulating theoretical models for, and the practical determination of numerical solutions to, engineering problems, the energy (or variational) principles of mechanics hold a position of central importance. In this book, we are concerned with developing these principles in their application to equilibrium stress and deformation analysis: the subject matter is developed in a careful manner from the treatment of discrete rigid body/spring systems onto bar and more sophisticated continuum stressing situations culminating in an introduction to the finite element method.

The book is directed at a senior undergraduate and first year graduate level of work so that the usual basic experience of statics and 'mechanics of materials' is taken for granted. However, in teaching advanced engineering students over a number of years, the author has learnt that a frequent cause of difficulty with some advanced theory is an imperfect recollection of elementary principles, studied quite some time previously, with a consequent difficulty in relating these principles to the current situation. This seems to be especially true of the energy principles of mechanics which often receive only a small amount of attention in undergraduate courses. Accordingly the subject matter is developed from basic principles to a good level in a reasonably self-contained manner so as to facilitate this process of reference back from advanced to elementary ideas. It is intended to be useful for individual study or as supplementary reading for a course. The presentation is such that the first two chapters alone could form the basis for a series of lectures to undergraduate mechanical or structural engineers; the remainder is written for the first year graduate student or practicing engineer who has cause to take up advanced methods in his professional work.

The principles of virtual work and virtual complementary work are concerned with describing the static equilibrium of a system by comparing the relative merits of a number of candidate solutions and identifying the true one by requiring that some property of the system shall attain a stationary value. Whilst the overtones of economy in nature implicit in this approach have considerable plausibility; it, and the concomitant notions of virtual displacements and virtual forces, are a source of unease to engineers who are usually more familiar with the cause and effect philosophy of vector mechanics. The earlier sections of the book are consequently devoted to overcoming this unease by providing a very full account of the energy principles in relation to discrete systems; in this way the underlying philosophy can be absorbed without the distractions of the higher level of mathematics required when continuum systems are discussed. It is only when these basic concepts have been thoroughly understood as a result of reflection and practice in treating relatively simple systems that one can approach more complex situations with confidence.

Chapter 1 is concerned only with discrete or lumped spring and rigid body systems. It begins with a brief review of some very basic notions in mechanics and continues with a careful discussion of system configuration in which constrained and generalised coordinates are introduced. For the purposes of contrast with the subsequently developed energy methods, the principles of vector statics are reviewed and it is shown that the solution to any problem in the statics of deformable bodies involves three essential ingredients (a) the consideration of equilibrium of forces, (b) the requirement of geometric fit, or compatibility of the displacements and (c) some force-deformation characteristic for the elements of the system. Work, energy, the idea of a virtual displacement, the principles of virtual work and stationary potential energy are then described and applied to a few simple situations. In a similar way, we then discuss and apply the ideas of complementary work and energy, statically independent forces and the principles of virtual complementary work and complementary energy. The presentation is such as to highlight the complementary nature of two characteristic approaches to problems in structural mechanics: on the one hand, the virtual work principle leads to the stiffness method of analysis whilst on the other the virtual complementary work principle leads to the flexibility method. We conclude with a discussion of the stability of equilibrium. The author considers this chapter to be of crucial importance in that all the essentials for a variational approach to mechanics problems are introduced; the remainder of the book is concerned with the refinements necessary to facilitate tackling problems of greater complexity.

In Chapter 2, we apply the previously established principles to structures comprised of bar-type elements. We study the characteristics of individual elements under conditions of axial load, torsion, bending and transverse shear and consider the idealised assemblages of pin jointed and rigid jointed frames. By means of these relatively straightforward systems we illustrate the freedom of choice one has for virtual displacements and virtual forces and show how the former can be used to evaluate the degree of statical indeterminancy of a given structure. The principles of virtual work and virtual complementary work are applied to such situations as will highlight their scope and limitations without undue computational effort and once more to formulate the alternative stiffness or flexibility methods of structural analysis.

Solid continuum mechanics is the body of analytic techniques by means of which a high level of precision has been attained in the stress analysis of many complex systems: Chapter 3 is concerned with the foundations of this topic. Whether one bases this level of treatment on vector or variational principles, it is necessary to clarify the concepts of a continuum, the state of stress and strain and to establish a constitutive relation between stress and strain for the continuum material: the earlier parts of the chapter are concerned with these matters. The interpretation of equilibrium and compatibility conditions at a continuum level are discussed and, with these foundations firmly lain down, the formulation of the general problem of elastic stress analysis follows. The chapter concludes with an account of some specific classes of problem.

When variational principles are applied to continuum problems, the appropriate mathematical tool is the calculus of variations. Chapter 4 is concerned with an account of its principles which is by no means exhaustive but which the author feels is adequate for the purposes of this book. Its central purpose is to show that when the behaviour of a continuous system is in accordance with some minimum principle, the procedures of the variational calculus yield differential equations together with appropriate boundary conditions describing the behaviour. The subject matter is essential preparation for Chapter 6.

The so-called direct methods of the calculus of variations provide powerful practical techniques for obtaining numerical solutions to very complex stressing problems of engineering significance. They comprise a body of approximation methods for treating continuum problems. Several such methods have been developed over the years, but in Chapter 5 we restrict ourselves to a discussion of the more widely used ones due to Rayleigh, Ritz, Kantorovitch, Galerkin, Euler and Trefftz. These methods are of importance in their own right, but they also form a valuable preparation for an understanding of the more modern finite element procedure.

Chapter 6 is concerned with formulating the variational principles of mechanics for a solid continuum. Here the ideas and methods of Chapters 3 and 4 are brought together in a generalisation of the procedures of Chapters 1 and 2 and Reissner's principle is introduced. We see how the relevant equations of equilibrium and compatibility can be deduced for an 'exact' theory and, by way of specific classes of problem, how all the relevant equations for an 'approximate' theory can be deduced in a systematic and consistent manner. A designer cannot be content with merely producing an adequate approximate theory to describe his current problem, he has to produce quantitative data from which suitable proportions may be determined for structural and machine parts. Then, since even an approximate theory may yield differential equations which are difficult to handle, the direct methods of the calculus of variations can be extremely valuable. Chapter 7 is a collection of substantial examples in which the methods of Chapter 5 are applied to a variety of situations of practical interest. The examples show that the effect of these approaches is to descretise a continuum model, that is the behaviour of a system with an infinite number of degrees of freedom is approximated by one having only a finite number. Under these circumstances the virtual work principle is concerned with determining a compatible configuration for which equilibrium is satisfied only in some average sense whilst the virtual complementary work principle is concerned with equilibriant stress systems for which compatibility is satisfied only in some average sense. Considerable emphasis is placed on clarifying these aspects of the work since confidence in applying and interpreting the results of these methods only comes from appreciating the physical implications of the various mathematical operations carried out.

The finite element method is introduced in Chapter 8. Emphasis is placed on the so-called displacement method and a conscious effort is made to relate the method to the material in the earlier sections of the book. Following a discussion on generating displacement models, a number of classes of problem are treated showing that the characteristic feature of the method is the generation of stiffness equilibrium equations in which the basic unknowns are displacements. The torsion problem, formulated in terms of the Prandtl stress function, is treated to provide an example of a complementary energy approach whilst the chapter is concluded by a brief introduction to the more sophisticated hybrid and mixed formulations which have a number of important applications.

Some useful results from mathematics are collected together in the Appendix.

The variational methods of mechanics are amongst the most beautiful of man's intellectual achievements and at the same time provide some of his most useful analytic tools. Their range of application stretches well beyond that covered here but to keep the book to a moderate length a number of interesting topics have had to be put aside. Thus, to add a satisfactory treatment of shell problems, plasticity or the stability and dynamics of elastic systems would require substantially more space; on the other hand, to replace the introduction to finite elements by, say, an account of stability or dynamics is largely a matter of personal preference. The object has been to produce a reasonably self contained developing account of the use of energy methods in determining the displacements and stresses in a mechanical system in a state of stable equilibrium. It is my hope that, with the pointers given in the Introduction, the reader who wishes to proceed to the literature dealing with the more specialist areas of this work will feel equipped so to do.

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Author's Introduction

In spite of the advanced stage to which Engineering Science has progressed, it is not yet capable of describing the behaviour of real engineering systems in every particular. Consequently, an essential element in the process of designing such systems is the generation of conceptual models (idealisations) of real situations to which one's current stock of analytic and numerical techniques may be applied. Depending on the requirements of the problem under consideration and the current state of one's capability for solution, models at various levels of sophistication may be contemplated and calculations based on these models will yield predictions at various levels of precision. Although one may hope that such calculations will be adequate for the purpose at hand, it is often the case (and *usually* so for advanced products) that the development of a satisfactory design requires laboratory work and in-service experience to confirm the adequacy of some theoretical model or to overcome some problem for which no theoretical treatment is contemplated.

In the design of a given product, one may need to refer to several branches of engineering science. Yet, whatever function the device must perform, it will need to be structurally sound. Now it is very obvious that if a product actually breaks it certainly will not have been adequate, but malfunction may also occur due to excessive distortion under load or to spurious response to some environmental disturbance such as temperature or vibration. Evidently stress and deformation analysis is central to design and in this book we are concerned with showing what a powerful aid to this work is offered by the variational principles of mechanics. The aim is to develop the subject in a self contained coherent manner from elementary beginnings to advanced applications. To this end, we first consider one of the most basic models of a mechanical system, a lumped spring/rigid body arrangement, and acquire an appreciation of the underlying notions of the variational approach without the distractions of the higher level of mathematics necessary for continuum situations. In this way, we gain a 'feel' for virtual displacements (variations to a configuration), virtual forces (variations to equilibrium states) and a sense of nature working with sublime economy of effort to achieve some end. In preparation for a more sophisticated treatment of systems with distributed mechanical properties, we proceed to clarify the basic ideas of continuum mechanics and to examine the variational calculus, which is the essential mathematical tool for manipulating energy principles applied to continua. With this ground work consolidated, it is possible fully to appreciate the elegance and practical utility of the variational approach and to benefit from the insight it provides. In formulating, by the vector approach, an approximate theory say for beams or plates, it is not always clear just what mathematical terms are significant and what are not: in contrast, we perceive that the variational approach yields a consistent statement of the boundary value problem automatically. Further, in the event of even the approximate theory being too difficult to handle analytically, a whole range of numerical processes, culminating in the finite element method, is available to us. Except for a brief reference to stability in Section 1.9, we shall be concerned with problems in *equilibrium* stress analysis; a consideration of large deformations, buckling, dynamics and plasticity or viscoelaticity would require far more space than we can afford. However, to provide some indication of how the energy methods may be applied to problem areas beyond those of immediate interest in this book, we shall devote the remainder of these introductory pages to a very brief discussion of just three beam problems. We shall discover that whilst the equilibrium configuration of a finite degree of freedom system can be described by a set of simultaneous algebraic equations,

$$[K]\{q\} = \{Q\} , \tag{I.1}$$

an analogous treatment of the buckling or free vibration of such systems leads to the eigenvalue problem

$$([K] - \lambda[M])\{q\} = 0 . \tag{I.2}$$

Suppose a beam, simply supported at both ends, carries a distributed load of intensity $p(x)$ per unit length over its entire length l. As we shall see in Chapter 1, the equilibrium configuration of a mechanical system may be identified by means of the principle of stationary total potential energy; then, in the case of this beam, the deflexion curve $w(x)$ is identified by

$$\delta V = 0 \tag{I.3}$$

where

$$V = \int_0^l \frac{EI}{2}(w'')^2\,dx - \int_0^l pw\,dx\,\dagger , \tag{I.4}$$

The appropriate mathematical tool for handling equation (I.3) when (I.4) is substituted into it is the Calculus of Variations (Chapter 4). If we follow its rules, we find that equation (I.3) leads to

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = p \tag{I.5}$$

together with appropriate boundary conditions; equation (I.5) is the familiar differential equation of equilibrium for a beam. If the bending stiffness EI is constant, solving equation (I.5) is a straightforward matter, but if EI is a function of position along the beam, the solution may be awkward to achieve. An alternative approach, which has increasing attraction as the systems of interest become more complex, is to seek an *approximate* solution by *assuming* a representation for the deflexion curve in the form of a series of prescribed functions $\phi(x)$.

† See Chapters 2 and 6.

Then,

$$w(x) = \sum_{i=1}^{n} q_i\phi_i(x) \quad , \tag{I.6}$$

where the q's are parameters to be determined. Substituting from equation (I.6) into (I.4),

$$V = \frac{1}{2}\sum_i \sum_j q_i q_j \int_0^l EI\phi_i''\phi_j''\mathrm{d}x - \sum_i q_i \int_0^l p\phi_i \mathrm{d}x \ . \tag{I.7}$$

Denoting
$$k_{ij} = \int_0^l EI\phi_i''\phi_j''\mathrm{d}x = k_{ji} \tag{I.8}$$

and
$$Q_i = \int_0^l p\phi_i\mathrm{d}x \quad , \tag{I.9}$$

equation (I.7) becomes

$$V = \frac{1}{2}\sum_i \sum_j k_{ij}q_iq_j - \sum_i Q_iq_i \ . \tag{I.10}$$

The reader who is familiar with matrix algebra† will recognize this as

$$V = \tfrac{1}{2}\{q\}^t[K]\{q\} - \{q\}^t\{Q\} \ . \tag{I.11}$$

Here, $[K]$ is a matrix of the **stiffness coefficients** k_{ij}, $\{q\}$ is a vector of **generalised coordinates** and $\{Q\}$ is a vector of **generalised forces** corresponding to the $\{q\}$ (see Chapter 1): we observe from equation (I.8) that $[K]$ is symmetric.

Equation (I.11) expresses the typical form of the potential energy expression for a finite degree of freedom system and we see that the energy approach has furnished, in a rational manner, the appropriate mechanical characteristic $[K]$ and disturbance $\{Q\}$ for a substitute discrete system whose behaviour approximates that of the real beam. (We will not dwell on the considerations affecting the choice of the ϕ's here, they are discussed in the main body of the text). $\delta V = 0$, applied to equation (I.11) now ensures satisfaction of beam equilibrium only in some average way summarised in

$$[K]\{q\} = \{Q\} \ . \tag{I.12}$$

Whilst the above beam takes up its bowed shape, its ends will travel towards each other by some small amount Δ. As this movement occurs, the moment arms of the support reactions change very slightly, but in the usual theory, where the deflexion is small, this effect is negligible. If, on the other hand, the beam also supports an axial thrust T, as shown in Fig. I.1, the bow in the beam causes T to have a moment arm (equal to the deflexion) and hence a bending action as well as the ordinary stress = force divided by area effect. This feature of an alteration in the action of a load as the deflexion occurs is a simple example of

† The reader who is not familiar with this notation will find a summary in the Appendix.

non-linear behaviour rooted in structural geometric effects rather than a non-linear stress-strain law for the material.

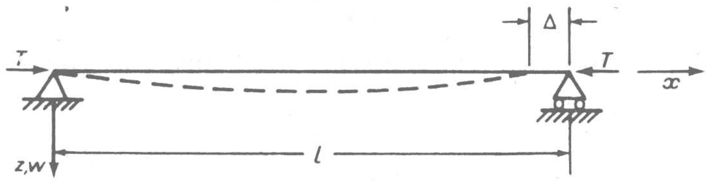

Fig. I.1

To apply the energy method to this problem, it is only necessary to take into account the work done by T in travelling through the distance Δ.† Then, the total potential energy of the system is

$$V = \int_0^l \frac{EI}{2}(w'')^2 dx - \int_0^l pw dx - T\Delta \quad . \tag{I.12}$$

We are concerned with bending actions as opposed to direct compression here so that the final length of the bowed beam is still l. Then,

$$l = \int_{x=0}^{x=(l-\Delta)} ds \quad . \tag{I.13}$$

Now
$$ds = \left[1 + \left(\frac{dw}{dx}\right)^2\right]^{1/2} dx \tag{I.14}$$

and, if we suppose dw/dx to be small, expanding by the binomial theorem and ignoring terms in $(dw/dx)^4$ etc, yields

$$ds \doteqdot \left[1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] dx \quad . \tag{I.15}$$

Substituting from equation (I.15) into (I.13), we have

$$\Delta = \frac{1}{2}\int_0^l (w')^2 dx \quad . \tag{I.16}$$

For equilibrium, $\delta V = 0$ and applying the formal processes of the calculus of variations to equation (I.12), we have the Euler-Lagrange equation for a uniform beam column as

$$EI\frac{d^4 w}{dx^4} + T\frac{d^2 w}{dx^2} = p \quad . \tag{I.17}$$

† Notice that Δ arises from the bow in the beam alone and not from direct compression. We have ignored this latter effect since we are interested in the buckling problem and it can be shown that the value of the buckling load is not affected by the value of the direct compression energy.

If the beam supports the end thrust alone, $p = 0$ and equation (I.17) becomes

$$EI\frac{d^4w}{dx^4} + T\frac{d^2w}{dx^2} = 0 \; . \qquad \text{(I.18)}$$

Equation (I.18) gives non-trivial solutions for w only for certain discrete values T_{crit} of the end thrust. These critical values of T are the **eigenvalues** of the problem and the corresponding solutions for w are the **eigenfunctions.** Equation (I.18) is a **buckling equation** and the smallest value of T_{crit} is the **buckling load.**

If we assume a solution for $w(x)$ in the form of a series of prescribed functions $\phi(x)$,

$$w(x) = \sum_{i=1}^{n} q_i\phi_i(x) \; , \qquad \text{(I.19)}$$

with the q_i parameters to be determined, the potential energy becomes

$$V = \frac{1}{2}\sum_i\sum_j q_iq_j\left(\int_0^l EI\phi_i''\phi_j''\mathrm{d}x\right) - \sum_i q_i\left(\int_0^l p\phi_i\mathrm{d}x\right) - \frac{1}{2}\sum_i\sum_j q_iq_j\left(\int_0^l T\phi_i'\phi_j'\mathrm{d}x\right) \; . \qquad \text{(I.20)}$$

Denoting

$$k_{ij} = \int_0^l EI\phi_i''\phi_j''\mathrm{d}x = k_{ji} \, , \quad s_{ij} = \int_0^l T\phi_i'\phi_j'\mathrm{d}x = s_{ji} \, ,$$

$$Q_i = \int_0^l p\phi_i\mathrm{d}x \, , \qquad \text{(I.21)}$$

k_{ij} is the usual stiffness coefficient with respect to q_i and q_j implicit in the assumed deflexion shape, s_{ij} is called the **geometric stiffness** coefficient and Q_i is a generalised force. Then,

$$V = \frac{1}{2}\sum_i\sum_j (k_{ij} - s_{ij})q_iq_j - \sum Q_iq_i \qquad \text{(I.22)}$$

or $\quad V = V(q\text{'s}) \; .$

Now, we shall see in Section 1.9 that the equilibrium configuration ceases to be stable when the second variation $\delta^2 V$ ceases to be positive definite. For the present example, this implies that:

$$|[K] - [S]| = 0 \; . \qquad \text{(I.23)}$$

This is an algebraic eigenvalue problem: expanding the determinant yields a polynomial in T, the roots of which are its critical values T_{crit}. The smallest value of T_{crit} is the **buckling load.** If we denote $T_{crit} = \lambda T$, where λ is a parameter, equation (I.23) can ' : written in the standard form

$$|[K] - \lambda[S]| = 0 \; . \qquad \text{(I.24)}$$

Equations (I.23) and (I.24) correspond to systems with a finite number of degrees of freedom and we see, once more, how an energy approach provides a convenient and rational means for calculating the appropriate parameters $[K]$ and $[S]$ for the discretised continuum.

It is often necessary to compute the response of a structure to dynamic loading: when a structure is set in motion, its elements experience accelerations so that inertia forces are called into play. According to **D'Alembert's principle**, a dynamics problem may be converted into a statics one by taking these inertia forces into account so that the principle of virtual work† may be employed. If the displacements are considered to be prescribed at two instants of time t_1 and t_2, so that any virtual displacements must vanish then, integrating the virtual work equation with respect to time, with the inertia forces incorporated into the body force term, leads to **Hamilton's principle**. This may be regarded as *the* basic principle of dynamics and states that:

Of all the geometrically possible motions which a system may execute, the true one is that which renders

$$\delta A = \delta \int_{t_1}^{t_2} L \, dt = 0 . \tag{I.25}$$

The quantity A is called the **Action**, whilst $L = T - U - \Omega$ is the **Langrangian** function. T is the kinetic energy, U the strain energy and Ω the potential energy of the applied loads.

Application of the formal processes of the calculus of variations to equation (I.25) for a continuum leads to partial differential equations of motion and appropriate boundary conditions. Alternatively, we may seek an approximate solution after the fashion we now employ in a further study of the beam.

Suppose the beam is excited by a distributed load $p(x, t)$, which is now a function of time and position, and that the deflected shape taken up can be approximated by

$$w(x, t) = \sum_{i=1}^{n} q_i(t)\phi_i(x) . \tag{I.26}$$

The ϕ's are prescribed bent shapes for the beam and so are functions of position x alone satisfying the boundary conditions, whilst the q's, which are functions of t, are the amplitudes of the constituent shapes making up $w(x, t)$.

If the mass per unit length of the beam is m, the kinetic energy is

$$T = \frac{1}{2}\int_0^l m(\dot{w})^2 dx = \frac{1}{2}\sum_i \sum_j \dot{q}_i \dot{q}_j \int_0^l m \, \phi_i \phi_j \, dx . \tag{I.27}$$

Denoting

$$m_{ij} = \int_0^l m \, \phi_i \phi_j \, dx = m_{ji} , \tag{I.28}$$

† See Chapter 1.

as elements of the mass matrix $[M]^\dagger$,

$$T = \tfrac{1}{2}\{\dot{q}\}^t[M]\{\dot{q}\} \quad . \tag{I.29}$$

The strain energy is given by

$$U = \tfrac{1}{2}\{q\}^t[K]\{q\} \quad , \tag{I.30}$$

where
$$k_{ij} = \int_0^l EI\phi_i''\phi_j''\mathrm{d}x$$

as before and
$$\Omega = -\{q\}^t\{Q\} \quad ,$$

with
$$Q_i(t) = \int_0^l p\phi_i\mathrm{d}x \quad . \tag{I.31}$$

Hamilton's principle now becomes

$$\delta\left[\int_{t_1}^{t_2}\left(\frac{1}{2}\{\dot{q}\}^t[M]\{\dot{q}\} - \frac{1}{2}\{q\}^t[K]\{q\} + \{q\}^t\{Q\}\right)\mathrm{d}t\right] = 0 \tag{I.32}$$

or
$$\int_{t_1}^{t_2}(\{\delta\dot{q}\}^t[M]\{\dot{q}\} - \{\delta q\}^t[K]\{q\} + \{\delta q\}^t\{Q\})\mathrm{d}t = 0 \tag{I.32a}$$

where we have taken advantage of the symmetry in $[M]$ and $[K]$. Integrating the first term by parts, $\dagger\dagger$

$$\text{First term} = |\{\delta q\}^t[M]\{\dot{q}\}|_{t_1}^{t_2} - \int_{t_1}^{t_2}\{\delta q\}^t[M]\{\ddot{q}\}\mathrm{d}t \quad . \tag{I.33}$$

But $\{\delta q\} = 0$ at t_1 and t_2 so that equation (I.32) becomes

$$\int_{t_1}^{t_2}\{\delta q\}^t([M]\{\ddot{q}\} + [K]\{q\} - \{Q\})\mathrm{d}t = 0 \quad . \tag{I.34}$$

At any instant, the $\{\delta q\}$ are arbitrary so that the motion is described by

$$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\} \quad . \tag{I.35}$$

The energy method has again proved to be a very convenient means of generating the significant form of the excitation force, $\{Q\}$, and the relevant system parameters $[M]$ and $[K]$.

$\dagger$ Notice that $[M]$ is not merely a diagonal matrix as it would be for a simple lumped parameter system.

$\dagger\dagger$ Alternatively, one could use **Lagrange's Equations**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

which also follow from Hamilton's principle.

When the beam is in a state of **Free Vibration,** $\{Q\} = \{0\}$ and $\{q\}$ is a harmonic function of time. Then,

$$w(x, t) = \sum_{i=1}^{n} \hat{q} \sin \omega t \, . \, \phi(x) \qquad (I.36)$$

Substitution in equation (I.35) yields

$$([K] - \lambda[M])\{\hat{q}\} = \{0\} \, , \qquad (I.37)$$

where $\lambda = \omega^2$. Non-trivial solutions $\{\hat{q}\}$ are obtained only when

$$|\,[K] - \lambda[M]\,| = 0 \, . \qquad (I.38)$$

The roots $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ of this polynomial correspond to the **Natural frequencies** of vibration of the discretised system whilst the corresponding $\{\hat{q}_1\}, \{\hat{q}_2\} \ldots$ are its **Natural modes.**

This book is concerned with the application of the variational principles of mechanics to the solution of problems in elastic stress analysis. In these introductory pages, the aim has been to give an impressionist view of the topics covered and to provide a few pointers as to how they may be extended to treat other problem areas which are significant in practical design. It is hoped that the student who has read this text and wishes to proceed to the literature dealing with these more specialist fields will feel equipped so to do.

Further reading

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