DETERMINISTIC OPERATIONS RESEARCH

Models and Methods in Linear Optimization

DAVID J. RADER JR.

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DETERMINISTIC OPERATIONS RESEARCH

To Cetta, Megan, and
Abby

PREFACE

This book represents the deterministic operations research course that I have taught to mathematics, computer science, and engineering students over the past 10 years. It deals with the study of linear optimization (both continuous and discrete), emphasizing the modeling of real problems as linear optimization problems and the designing of algorithms to solve them, covering topics in linear programming, network optimization, and integer programming.

In this book, I emphasize three aspects of deterministic operations research:

1. Modeling real-world problems as linear optimization problems.
2. Designing algorithms (both heuristic and exact methods) to solve these problems.
3. Using mathematical theory to improve our understanding of the problem, to improve existing algorithms, and to design new algorithms.

Although these aspects are important for both researchers and practitioners of operations research, they are not always in the forefront of operations research textbooks. It is true that many books highlight optimization modeling and algorithms to solve these problems; however, very few, if any, explicitly discuss the algorithm design process used to solve problems. This book is intended to fill that gap.

My intended audience for this book is a junior/senior mathematics course in deterministic operations research. It will serve as both comprehensive text for such a course and a reference book for future exploration of the subjects. The book can also be used for operations research courses in engineering departments and for first-year graduate courses in linear optimization. It is not meant to be a reference book for researchers, but to be a book for people wanting to learn about linear optimization problems and techniques. A student using this book should have been exposed to (multivariable) calculus and linear algebra, whenever possible.

I introduce concepts and approaches by mixing in many examples to both demonstrate and motivate theoretical concepts and then proving these results. Advanced ideas are described in a common-sense style that illustrates the core aspects of the idea in ways that are easy to understand, motivates the reader to understand how to think about the problem, not just what to think, and illustrates this using concrete examples. Each chapter of the book is designed to be the continuation of the "story" of how to both model and solve optimization problems by using the specific problems (linear and integer programs) as guides. This enables the reader (and instructors) to see how solution methods can be derived instead of just seeing the final product (the algorithms themselves).

I begin this book with an introduction to linear optimization modeling, incorporating many common modeling constructs through examples. In class I typically take about 20–25% of the term to discuss various models and the modeling process, so it is no surprise that the modeling chapters encompass that much of the book. I feel it is important for students to be able to translate a real-world situation into a mathematical language (here as an optimization model), be able to solve it using an optimization package, and be able to interpret the results. It is also important for students to see that there is a difference between the model and the data, and I try to emphasize this by discussing the general form of various constructs. I end my discussion of modeling by illustrating some common operations research problems and how they are solved. In addition, the exercises discuss various applications that have appeared in the research literature over the past 20 years.

Once the importance and usefulness of linear and integer programming models have been illustrated, I take a different approach to solution methods than most books. Typically, at an introductory level, solution methods for optimization problems are presented in the following style: (a) the problem is defined, (b) a solution technique is described and illustrated, and (c) the solution method is proven correct (it gives the correct answer) and various properties are studied. I illustrate how to design methods to solve these problems by first introducing the algorithm design paradigm through the use of heuristic methods for common integer programming models and then discussing general optimization approaches and showing how to use the properties of linear programs to specialize these general methods. This illustrates the algorithm design process.

I have found that the algorithm design process (which is key to deterministic operations research) can best be introduced through heuristic methods. This allows us to concentrate on the design itself without having to worry about finding the optimal solution. Once we have designed a few simple algorithms we are ready to search for optimal solutions by first learning through an example what properties an optimal solution must have and then designing (and improving) an algorithm to satisfy them.

One challenge facing instructors is the lack of a formal background in the subject, especially for mathematics professors. Many may not be familiar with how algorithms are designed, the importance of modeling and proper model formulation, and how models and the algorithms that solve them interact. This book will help such instructors understand these ideas and present them to the students. The book emphasizes the thought process involved with modeling optimization problems and both the design and the improvement of algorithms to solve these problems. Each chapter builds upon the previous ones to illustrate these different aspects. For example, by the time the reader is formally introduced to the simplex method (the primary algorithm for solving linear programs), we will have discussed how this algorithm could have been developed, giving the reader "ownership" of the algorithm.

This book is organized around the three aspects of deterministic operations research that I want to emphasize. The first four chapters discuss an introduction to operations research and optimization modeling. Chapters 5 through 8 discuss algorithm design for (continuous linear) optimization problems, culminating in the formal discussion of the simplex method. This includes an introduction to continuous optimization algorithms, convexity, Farkas' lemma, and the algebraic and geometric study of polyhedra. All of this culminates in the simplex method. Many will find it unusual to see the simplex method appear as late as Chapter 8 in this book. This is by design; I want to emphasize the importance of modeling first (and hence there are four chapters on models) and the algorithm design process next. This means that the simplex method, the end result of the design process, is important but not the sole focus of the book. In fact, because it is central to much of what we discuss in the book, it is symbolic that the simplex method is found in the middle chapter.

Chapters 9 through 11 discuss the duality theory of linear programs and the algorithmic and modeling ramifications of the theory. I also discuss Lagrangian duality and briefly mention optimality conditions for general nonlinear programs. In Chapters 10 and 11, I cover uses of duality, including sensitivity analysis, parametric programming, and such algorithmic uses as dual simplex, column generation, Dantzig–Wolfe decomposition, and the primal–dual interior point algorithm. Each of these topics highlights how theoretical results can be used to improve how we solve problems computationally.

Chapters 12 through 15 continue our algorithmic design theme by discussing network optimization and integer programming methods. I have incorporated some topics not typically given in introductory books, such as various label-correcting algorithms for the shortest path problem (as opposed to just Dykstra's method), the importance of preprocessing and probing in integer programming, cover inequalities for 0–1 knapsack constraints, lifting of valid inequalities, and branch-and-cut algorithms. In fact, the integer programming material in Chapter 14 introduces methods that are being implemented in software packages for integer programs. I finish the book with a discussion of modern metaheuristic techniques for discrete optimization problems. The methods form the backbone of many heuristic approaches to hard discrete problems.

Supplemental material will be key to the course, which will be placed on a dedicated web site http://www.rose-hulman.edu/~rader/DOR for the book. This includes the appropriate models (in various modeling languages) for each example in Chapters 2, 3, and 4, as well as Maple and Matlab content for many calculations used in the book. This will enable students and instructors to learn the concepts being introduced without needing to worry about software implementations. A collection of solutions to exercises is available to students; in addition, a full solutions manual will be made available for instructors.

The material from this book can be covered entirely over a two-semester sequence of courses. The core material of this book is contained in Chapters 5–9. Any course that includes the Simplex method (Chapter 8) needs Chapters 5–7 as background material. When I teach a course during one term using the material from this book, I have covered the following chapters over 40 class meetings (50 minutes each): Chapters 1, 2, 3, 5, 6, 7, 8, 9, 11.1–11.3, 13, and 14.1–14.5. In this course, I spend 8–10 lectures on material from Chapters 2 and 3 (optimization modeling); this is due to my personal bias toward the importance of optimization modeling, and users of this book would not need to spend as much time on these topics. I have included a more detailed topics list on the web site.

I have added various topics to this book that are optional materials to cover in a class, but provide the reader with more advanced topics for future study. These include (but are not limited to) sections on Farkas' lemma (Sections 6.3 and 9.5), Lagrangian duality (Section 9.7), parametric programming (Section 10.4), Dantzig–Wolfe decomposition (Section 11.3), primal–dual interior point algorithm (Section 11.6), network optimization algorithms (Chapter 12), and valid inequalities for 0–1 linear constraints and branch-and-cut algorithms (Sections 14.5 and 14.6).

It is impossible to write a book without any help or guidance, and that is especially true of this book. I want to thank John Wiley & Sons for publishing this book and to acknowledge in particular my editors Susanne Steitz-Filler and Jacqueline Palmieri for their assistance and guidance throughout this process.

I also want to acknowledge the comments I have received from colleagues Susan Martonosi and Kevin Hutson and other anonymous reviewers regarding the topics presented and their order and from the 10 years' worth of students at Rose-Hulman Institute of Technology who have used early versions of this manuscript as their primary material and have pointed out errors and omissions. Any errors that remain are of course my own.

I especially want to acknowledge Diane Evans, Richard Forrester, Allen Holder, and Mike Spivey, who have used this book in their classes. They have each provided me with numerous suggestions and comments on how to improve the presentation of material. Our conversations regarding the material has been immeasurable, and I cannot thank them enough for their contributions.

My colleagues in the Mathematics Department at Rose-Hulman Institute of Technology have been very supportive during the time I wrote this book. I particularly want to thank S. Allen Broughton, Ralph P. Grimaldi, and Jeffery J. Leader for their support and advice.

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David J. Rader, Jr.

Terre Haute, Indiana
April 2010

CONTENTS

PREFACE **xi**

1 INTRODUCTION TO OPERATIONS RESEARCH **1**

1.1 What is Deterministic Operations Research? 1
1.2 Introduction to Optimization Modeling 5
1.3 Common Classes of Mathematical Programs 13
1.4 About this Book 18
Exercises 19

2 LINEAR PROGRAMMING MODELING **21**

2.1 Resource Allocation Models 21
2.2 Work Scheduling Models 25
2.3 Models and Data 28
2.4 Blending Models 30
2.5 Production Process Models 36
2.6 Multiperiod Models: Work Scheduling and Inventory 42
2.7 Linearization of Special Nonlinear Models 46
2.8 Various Forms of Linear Programs 51
2.9 Network Models 56
Exercises 68

3 INTEGER AND COMBINATORIAL MODELS **85**

3.1 Fixed-Charge Models 85
3.2 Set Covering Models 92
3.3 Models Using Logical Constraints 94
3.4 Combinatorial Models 99
3.5 Sports Scheduling and an Introduction to IP Solution Techniques 109
Exercises 112

4 REAL-WORLD OPERATIONS RESEARCH APPLICATIONS: AN INTRODUCTION **126**

4.1 Vehicle Routing Problems 126
4.2 Facility Location and Network Design Models 130
4.3 Applications in the Airline Industry 139
Exercises 152

5 INTRODUCTION TO ALGORITHM DESIGN **159**

5.1 Exact and Heuristic Algorithms 159
5.2 What to Ask When Designing Algorithms? 161
5.3 Constructive versus Local Search Algorithms 162
5.4 How Good is our Heuristic Solution? 168
5.5 Examples of Constructive Methods 169
5.6 Example of a Local Search Method 181
5.7 Other Heuristic Methods 182
5.8 Designing Exact Methods: Optimality Conditions 183
Exercises 189

6 IMPROVING SEARCH ALGORITHMS AND CONVEXITY **197**

6.1 Improving Search and Optimal Solutions 198
6.2 Finding Better Solutions 201
6.3 Convexity: When Does Improving Search Imply Global Optimality? 214
6.4 Farkas' Lemma: When Can No Improving Feasible Direction be Found? 227
Exercises 232

7 GEOMETRY AND ALGEBRA OF LINEAR PROGRAMS **238**

7.1 Geometry and Algebra of "Corner Points" 239
7.2 Fundamental Theorem of Linear Programming 248
7.3 Linear Programs in Canonical Form 250
Exercises 262

8 SOLVING LINEAR PROGRAMS: SIMPLEX METHOD 271

8.1 Simplex Method 271
8.2 Making the Simplex Method More Efficient 285
8.3 Convergence, Degeneracy, and the Simplex Method 289
8.4 Finding an Initial Solution: Two-Phase Method 294
8.5 Bounded Simplex Method 300
8.6 Computational Issues 305
Exercises 308

9 LINEAR PROGRAMMING DUALITY 317

9.1 Motivation: Generating Bounds 317
9.2 Dual Linear Program 321
9.3 Duality Theorems 327
9.4 Another Interpretation of the Simplex Method 333
9.5 Farkas' Lemma Revisited 334
9.6 Economic Interpretation of the Dual 336
9.7 Another Duality Approach: Lagrangian Duality 338
Exercises 344

10 SENSITIVITY ANALYSIS OF LINEAR PROGRAMS 351

10.1 Graphical Sensitivity Analysis 351
10.2 Sensitivity Analysis Calculations 359
10.3 Use of Sensitivity Analysis 368
10.4 Parametric Programming 375
Exercises 380

11 ALGORITHMIC APPLICATIONS OF DUALITY 389

11.1 Dual Simplex Method 389
11.2 Transportation Problem 401
11.3 Column Generation 414
11.4 Dantzig–Wolfe Decomposition 420
11.5 Primal–Dual Interior Point Method 432
Exercises 441

12 NETWORK OPTIMIZATION ALGORITHMS 449

12.1 Introduction to Network Optimization 449
12.2 Shortest Path Problems 449
12.3 Maximum Flow Problems 458
12.4 Minimum Cost Network Flow Problems 470
Exercises 484

13 INTRODUCTION TO INTEGER PROGRAMMING 489

13.1 Basic Definitions and Formulations 490
13.2 Relaxations and Bounds 496
13.3 Preprocessing and Probing 500
13.4 When are Integer Programs "Easy?" 506
Exercises 511

14 SOLVING INTEGER PROGRAMS: EXACT METHODS 514

14.1 Complete Enumeration 514
14.2 Branch-and-Bound Methods 516
14.3 Valid Inequalities and Cutting Planes 524
14.4 Gomory's Cutting Plane Algorithm 531
14.5 Valid Inequalities for 0–1 Knapsack Constraints 536
14.6 Branch-and-Cut Algorithms 540
14.7 Computational Issues 547
Exercises 551

15 SOLVING INTEGER PROGRAMS: MODERN HEURISTIC TECHNIQUES 556

15.1 Review of Local Search Methods: Pros and Cons 556
15.2 Simulated Annealing 558
15.3 Tabu Search 562
15.4 Genetic Algorithms 565
15.5 GRASP Algorithms 570
Exercises 574

APPENDIX A BACKGROUND REVIEW 579

A.1 Basic Notation 579
A.2 Graph Theory 581
A.3 Linear Algebra 583

REFERENCE 597

INDEX 603

CHAPTER 1

INTRODUCTION TO OPERATIONS RESEARCH

1.1 WHAT IS DETERMINISTIC OPERATIONS RESEARCH?

Today, many "operations" problems are routinely solved throughout business and industry. These include determining work shifts for large departments, designing production facilities to maximize the throughput, allocating scarce resources (such as raw materials and labor) to meet demands at a minimum cost, or determining which investments to place available funds. Such problems are very different from the traditional problems that students of mathematics, science, and engineering often face, because there are no physical laws that can be used; for example, knowing how to use Newton's second law or Ohm's law will not help solve these industrial problems. To handle such problems, different **mathematical models** must be used.

> **Mathematical Model** A mathematical model is a collection of variables and the relationships needed to describe important features of a given problem.

Until World War II, most businesses and industries did not worry about such operations problems. This was partly because no formal mathematical discipline directly handled these problems, and partly because there was no urgent need for them. During World War II, military planners began working with scientists and mathematicians in order to apply a scientific approach to the management of the war effort, in which they began to devise mathematical models to deal with such issues. After the war, others began to look at these techniques for industry, which brought about the beginning of **Operations Research**.

> **Operations Research** Operations research (OR) is the study of how to form mathematical models of complex science, engineering, industrial, and management problems and how to analyze them using mathematical techniques.

Operations research is the name most often used to describe the mathematical study of such problems. In business, it is also known as *Management Science* or *Operations Management.* Typically, OR is divided into two areas: *deterministic operations research*, where all parameters are fixed, and *stochastic operations research*, where we assume some of the problem parameters are random. This book deals exclusively with deterministic operations research.

There have been many successful and interesting applications of operations research techniques since the 1940s. In fact, operations research has been used to solve many diverse problems.

1. *Scheduling ACC Basketball.* Nemhauser and Trick [67] discussed how they used OR techniques to determine successful schedules for the Atlantic Coast Conference basketball season, which consists of a round-robin schedule among 10 teams stretching from Maryland to Florida.
2. *Designing Communication Networks.* Balakrishnan, Magnanti, and Mirchandani [4] look at the problem of designing survivable networks with high bandwidth in order to minimize costs. These problems have great importance due to the explosion of phone, cable, and video services.
3. *Scheduled Aircraft Maintenance.* Gopalan and Talluri [51] studied how to schedule aircraft maintenance when there are many operating restrictions in place, such as making sure the aircraft is at the correct maintenance location within a certain amount of time.
4. *Truck Dispatching for Oil Pickup.* Bixby and Lee [16] look at the vehicle routing and scheduling problem that arise from an oil company's desire to better utilize truck fleets and drivers.
5. *Component Assignment in Printed Circuit Board Manufacturing.* Hillier and Brandeau [58] consider the problem of determining the minimum cost assignment of various components to insertion machines that are used in the automated assembly of printed circuit boards.

OR is much more than the formation of mathematical models of problems; it also deals with the solution of these models and the evaluation of their solutions with regard to the real-world problem. In deterministic OR models, we typically formulate the problem as an optimization problem, where we want to minimize or maximize a function (such as costs, profits, time, etc.) given that the solution must satisfy certain restrictions and requirements (demand met, scarce resources, etc.). Such optimization problems are often referred to as **Mathematical Programs**.

> **Mathematical Program** A mathematical program or *Optimization Model* is a mathematical structure where problem choices are represented by **decision variables**. Using these decision variables, mathematical programs look to optimize some **objective function** of the decision variables subject to certain **constraints** that limit the possible choices (values of decision variables). It can be written in the most general form (assuming maximization problem)
>
> $$\max \{f(\mathbf{x}) : \mathbf{x} \in S\},$$
>
> where $\mathbf{x}$ is a vector of decision variables, $f(\mathbf{x})$ is the objective function, and the set S is the set of values for the decision variables satisfying all of the constraints.

The three highlighted terms in the above definition, decision variables, objective function, and constraints, represent the fundamental concerns of any OR model. In fact, it is the objective function that makes a model an optimization model. This was fairly revolutionary in 1947, when George Dantzig used an objective function to determine which decisions were better than others. Before that, there was not much thought given in modeling toward objective functions; instead, basic ground rules given by those in authority were used to guide the search for the "best answer."[1]

But how do we create a mathematical model? That is the real question in any study of operations research. The whole modeling process involves a four-stage cycle:

1. *Problem Definition and Data Gathering.* Here is where the real work begins. Actually knowing what problem we wish to model is more important than most people realize. If we have no idea what we want to determine, how do we know if we have a reasonable solution or not? Next, a modeler must observe the system and collect data to estimate any parameters of the model.
2. *Creating the Mathematical Model.* Once the problem is defined and data have been collected, it is now time to develop a model of the situation. This involves determining decision variables, constraints, and objectives that are needed in the problem. Sometimes it involves only determining which formula we need to use. Models can be either deterministic, as studied in this book, or stochastic, in which case our data are more important because we need them to verify assumptions on probability distributions, means, and so on.
3. *Solving the Mathematical Model.* This can have many different forms. If we have a deterministic optimization model, we try to find the values of the

[1] Dantzig reminisces about this in Ref. [26].