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PREFACE

This book contains a revised and expanded version of the lecture notes of two seminar series given during the academic year 1976/77 at the Department of Mathematics and Statistics of the University of Calgary, and in the summer of 1978 at the Institute of Theoretical Physics of the Technical University Clausthal. The aim of the seminars was to present geometric quantization from the point of view of its applications to quantum mechanics, and to introduce the quantum dynamics of various physical systems as the result of the geometric quantization of the classical dynamics of these systems.

The group representation aspects of geometric quantization as well as proofs of the existence and the uniqueness of the introduced structures can be found in the expository papers of Blattner, Kostant, Sternberg and Wolf, and also in the references quoted in these papers. The books of Souriau (1970) and Simms and Woodhouse (1976) present the theory of geometric quantization and its relationship to quantum mechanics. The purpose of the present book is to complement the preceding ones by including new developments of the theory and emphasizing the computations leading to results in quantum mechanics.

I am greatly indebted to the participants of the seminars, in particular John Baxter, Eugene Couch, Jan Tarski, and Peter Zvengrowski, for encouragement and enlightening discussions, and to Bertram Kostant, John Rawnsley and David Simms for their interest in this work and their very helpful suggestions. Special thanks are due to Liisa Heikkila, Pat Dalgetty, and Katherine MacDougall for careful and patient typing and retyping of the manuscript, and to Mark Gotay for editing the final draft of this work.

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Calgary, September, 1979 Jędrzej Śniatycki

CONTENTS

1. INTRODUCTION

1.1. Background

A classical system is described by the Poisson algebra of functions on the phase space of the system. Quantization associates to each classical system a Hilbert space $\mathscr{H}$ of quantum states and defines a map $\mathscr{Q}$ from a subset of the Poisson algebra to the space of symmetric operators on $\mathscr{H}$. The domain of $\mathscr{Q}$ consists of all "$\mathscr{Q}$-quantizable" functions. The definition of $\mathscr{Q}$ requires some additional structure on the phase space. The functions which generate one-parameter groups of canonical transformations preserving this additional structure are $\mathscr{Q}$-quantizable. They form a subalgebra of the Poisson algebra satisfying

$$[\mathscr{Q}f_1, \mathscr{Q}f_2] = i\hbar\mathscr{Q}[f_1, f_2],$$

where $[f_1, f_2]$ denotes the Poisson bracket of f_1 and f_2.

Two quantizations $\mathscr{Q}$ and $\mathscr{Q}'$ of the same classical system are *equivalent* if the domains of $\mathscr{Q}$ and $\mathscr{Q}'$ coincide and there exists a unitary operator $\mathscr{U}$ between the corresponding representation spaces such that, for each quantizable function f,

$$\mathscr{U}(\mathscr{Q}f) = (\mathscr{Q}'f)\mathscr{U} .$$

In physics, one is not interested in the whole Poisson algebra but rather in its subset consisting of functions with a definite physical interpretation, e.g., energy, momentum, and so on. Therefore, one may weaken the notion of equivalence of quantizations by requiring only that the physically interesting functions be contained in the intersection of the domains of $\mathscr{Q}$ and $\mathscr{Q}'$, and that the operator $\mathscr{U}$ intertwine the quantizations of these functions. This weaker notion of equivalence depends very much on our knowledge of the physical system under consideration and our judgement as to which functions are physically important.

There is a striking similarity between the canonical quantization of classical systems and the orbit method of construction of irreducible unitary representations of Lie groups. This similarity was recognized by Kostant, who wrote in the introduction of his 1970 paper entitled "Quantization and Unitary Representations":

> . . . We have found that when the notion of what the physicists mean by quantizing a function is suitably generalized and made rigorous, one may develop a theory which goes a long way towards constructing all the irreducible unitary representations of a connected Lie group. In the compact case it encompasses the Borel-Weil theorem. Generalizing Kirillov's result on nilpotent groups, L. Auslander and I have shown that it yields all the irreducible unitary representations of a solvable group of type I. (Also a

criterion for being of type I is simply expressed in terms of the theory.) For the semi-simple case, by results of Harish-Chandra and Schmid, it appears that enough representations are constructed this way to decompose the regular representation.

The geometric formulation of the canonical quantization scheme in physics was studied independently by Souriau. A comprehensive presentation of Souriau's theory of geometric quantization is contained in his book entitled "Structure des Systèmes Dynamiques" published in 1970. The works of Kostant and Souriau are the sources of the geometric quantization theory, also referred to as the "Kostant-Souriau theory."

The next fundamental development of the geometric quantization theory was due to Blattner, Kostant, and Sternberg [cf. Blattner (1973)]. It consists of the construction of a sesquilinear pairing between the representation spaces of the same classical system, usually referred to as a "Blattner-Kostant-Sternberg kernel." In some cases the pairing leads to the operator $\mathscr{U}$ intertwining the quantizations. As a result, one obtains a larger class of quantizable functions and the means of studying the equivalence of quantizations.

Geometric quantization is essentially a globalization of the canonical quantization scheme in which the additional structure needed for quantization is explicitly expressed in geometric terms. The theory, only about a decade old, is at a preliminary stage of its development. At present, it provides a unified framework for the quantization of classical systems which, when applied to most classical systems of physical interest, yields the expected quantum theories for these systems and removes some of the ambiguities left by other quantization schemes. It enables us to pose questions about the quantum theories corresponding to a given classical system and gives some partial answers. However, many issues remain unresolved. Among them are basic questions about the structure of the representation space, the search for appropriate conditions guaranteeing the convergence of the integrals involved in the Blattner-Kostant-Sternberg kernels and the unitarity of the intertwining operators defined by these kernels, etc. On a more specific level, there are cases when the geometric quantization of functions of physical interest poses such technical or theoretical difficulties that the corresponding quantum operators remain ambiguous. Some of these problems will be solved within the framework of the present theory. The others might require a modification of the theory; there are already indications that some modifications of the theory are inevitable.

The aim of this book is to present the theory of geometric quantization from the point of view of its applications to quantum mechanics, and to introduce the quantum dynamics of various physical systems as the result of the geometric quantization of the classical dynamics of these systems. It is assumed that the reader is familiar with classical and quantum mechanics and with the geometry of manifolds including the theory of connections. The proofs of the existence and the uniqueness of the structures introduced are omitted. On the other hand, all of the basic steps involved in computations are given, even though they may involve standard techniques.

A chapter by chapter description of the contents of the book follows.

1.2. Hamiltonian dynamics

A comprehensive exposition of classical mechanics containing references to the original papers is given by Whittaker (1961). The modern differential geometric approach adopted here follows Abraham and Marsden (1978).

The *phase space* of a dynamical system is a smooth manifold X endowed with a symplectic form ω defined by the Lagrange bracket. To each smooth function f on X, there is associated the *Hamiltonian vector field* ξ_f of f, defined by

$$\xi_f \lrcorner \omega = -df,$$

as well as the one-parameter group $\phi_f{}^t$ of canonical transformations of (X,ω) generated by f which is obtained by integrating the vector field ξ_f. Define local coordinates $(q^1,\ldots,q^n,p_1,\ldots,p_n)$ on X, where $n = \frac{1}{2}\dim X$, such that

$$\omega = \sum_i dp_i \wedge dq^i.$$

In such a "canonical" chart, the integral curves of ξ_f satisfy the canonical equations of Hamilton with the Hamiltonian f.

The mapping $f \mapsto \xi_f$ pulls back the Lie algebra structure from the space of smooth vector fields on X to the space of smooth functions on X. The space of smooth functions on X with this induced Lie algebra structure is called the *Poisson algebra* of (X,ω).

The Hamiltonian formulation can be extended to relativistic dynamics. The Hamiltonian vector field of the mass-squared function yields the covariant form of the equations of motion. The interaction with an external electromagnetic field f is taken into account by adding the term ef to the symplectic form, where e is the charge of the particle. This approach to the relativistic dynamics of a charged particle is due to Souriau (1970). It has the advantage that it enables one to discuss the Hamiltonian dynamics of a relativistic charged particle without any reference to the electromagnetic potentials.

The evolution space formulation of Newtonian dynamics is due to Lichnerowicz (1943). For time-dependent dynamics, the evolution space formulation is more appropriate than the phase space formulation which requires a time-dependent Hamiltonian. The evolution space formulation of single particle dynamics is given following Śniatycki and Tulczyjew (1972); see also Souriau (1970).

1.3. Prequantization

In the first step of geometric quantization one associates, to each smooth function f on X, a linear operator $\mathscr{P}f$ such that $\mathscr{P}1$ is the identity operator and

$$[\mathscr{P}f, \mathscr{P}g] = i\hbar\,\mathscr{P}[f,g].$$

This is done by introducing a complex line bundle L over X with a connection ∇ and an invariant Hermitian form <,> such that

$$\text{curvature } \nabla = -\hbar^{-1}\omega.$$

Such a line bundle exists if and only if $\hbar^{-1}\omega$ defines an integral de Rham cohomology class. This condition, referred to as the *prequantization condition*, gives rise to the quantization of charge in Sec. 10.1 and spin in Sec. 11.2. The operators $\mathcal{P}f$ act on the space of sections of L as follows. The one-parameter group ϕ_f^t of canonical transformations generated by f has a unique lift to a one-parameter group of connection preserving transformations of L which defines the action of ϕ_f^t on the space of sections of L. The operator $\mathcal{P}f$ is then defined by

$$\mathcal{P}f[\lambda] = i\hbar \frac{d}{dt}(\phi_f^t \lambda)\Big|_{t=0}.$$

This definition also makes sense if f defines only a local one-parameter group of local canonical transformations.

For a function f on X such that the Hamiltonian vector field ξ_f is complete, the one-parameter group of linear transformations $\lambda \mapsto \phi_f^t$ preserves the scalar product given by

$$\langle\lambda_1|\lambda_2\rangle = \int_X \langle\lambda_1,\lambda_2\rangle\, \omega^n.$$

Hence, the operator $\mathcal{P}f$, defined originally on smooth sections of L, extends to a self-adjoint operator on the Hilbert space of square integrable sections of L. However, if we wanted to give a probabilistic interpretation to the scalar product by associating to $\langle\lambda,\lambda\rangle(x)$ the probability density of finding the "quantum" state described by λ in the classical state described by the point x in the phase space X, we would violate the uncertainty principle since square integrable sections of L can have arbitrarily small support. The space of all square integrable sections of L is too "big" to serve as the space of wave functions.

The prequantization of symplectic manifolds has been studied independently by Kostant (1970a) and Souriau (1970). Some physical implications of prequantization are discussed by Elhadad (1974), Kostant (1972), Rawnsley (1972; 1974), Renuard (1969), Simms (1972, 1973a,b), Souriau (1970), Streater (1967), and Śniatycki (1974). See also Sławianowski (1971, 1972) and Weinstein (1973).

The formulation of the theory of connections in complex line bundles given in Sec. 3.1 follows the general theory of connections given in Kobayashi and Nomizu (1963), modified by the identification of the complex line bundle without the zero section with the associated principal fibre bundle. The presentation of prequantization given in Sections 3.2 and 3.3 follows essentially the exposition of Kostant (1970a), where one may find the proofs of the theorems regarding the existence and the uniqueness of the prequantization structures.

1.4. Representation space

In order to reduce the prequantization representation one has to introduce a classical counterpart of a complete set of commuting observables. A first choice would be a set of n independent functions $f_1, \ldots, f_n$ on X satisfying

$$[f_i, f_j] = 0 \quad \text{for} \quad i,j = 1,2,\ldots,n$$

such that their Hamiltonian vector fields are complete. The complex linear combinations of the Hamiltonian vector fields $\xi_{f_1}, \ldots, \xi_{f_n}$ give rise to a complex distribution F on X such that

$$[F,F] \subseteq F$$
$$\dim_{\mathbf{C}} F = \tfrac{1}{2} \dim X$$
$$\omega | F \times F = 0.$$

For many phase spaces of interest there does not exist such a set of functions. If one drops the assumption that the f_i be real and globally defined one is led to the notion of a *polarization* of (X,ω), that is, a complex distribution F on X satisfying the conditions given above. For technical reasons we assume that

$$D = F \cap \overline{F} \cap \mathcal{T}X$$

and

$$E = (F + \overline{F}) \cap \mathcal{T}X$$

are involutive distributions on X, and that the spaces X/D and X/E of the integral manifolds of D and E, respectively, are quotient manifolds of X with projections π_D and π_E. A polarization F satisfying these additional conditions is called *strongly admissible*.

Given a polarization F of (X,ω), one could take the space of sections of the prequantization line bundle L which are covariantly constant along F to form the representation space. However, if λ_1 and λ_2 are sections of L covariantly constant along F, their Hermitian product $\langle\lambda_1,\lambda_2\rangle$ is a function constant along D and its integral over X diverges unless the leaves of D are compact. Thus, we should integrate $\langle\lambda_1,\lambda_2\rangle$ over X/D, but we do not have a natural measure on X/D. In order to circumvent this difficulty, one introduces a bundle $\sqrt{\wedge^n F}$ *sections of which can be paired to* yield densities on X/D. The bundle $\sqrt{\wedge^n F}$ leads also to