CLASSIC REVIEWS IN MATHEMATICS AND MATHEMATICAL PHYSICS

Topological and Algebraic Geometry Methods in Contemporary Mathematical Physics

B. A. Dubrovin, I. M. Krichever and S. P. Novikov

Topological and Algebraic Geometry Methods in Contemporary Mathematical Physics

B. A. Dubrovin

International School for Advanced Studies, Trieste, Italy

I. M. Krichever

L. D. Landau Institute of Theoretical Physics, Russian Academy of Sciences, Moscow

and

S. P. Novikov

L. D. Landau Institute of Theoretical Physics, Russian Academy of Sciences, Moscow

C S P

First published in Soviet Scientific Reviews Section C © 1998 OPA (Overseas Publishers Association) N.V. Published by license under the Harwood Academic Publishers imprint.

Typeset by Siva Math Setters, Chennai, India
Printed and bound in Great Britain by Antony Rowe Ltd, Chippenham and Eastbourne

British Library Cataloguing in Publication Data
A catalogue record for this book has been requested

Library of Congress Cataloging in Publication Data
A catalogue record has been requested

ISBN 1-904868-30-4

ISSN 1028-0340

Cambridge Scientific Publishers Ltd
PO Box 806
Cottenham, Cambridge CB4 8QY
UK
www.cambridgescientificpublishers.com

Topological and Algebraic Geometry Methods in Contemporary Mathematical Physics

In the course of the last twenty years the methods of homotopy topology as well as those of the algebraic geometry of Riemann surfaces, Abelian varieties, and the related theta functions have begun to be used extensively in various branches of contemporary theoretical and mathematical physics. As examples, we mention that in the theory of so-called "liquid crystals", superfluid ^{3}He, Yang-Mills gauge fields, etc., many important solutions of the nonlinear equations arising in these theories have been found, which have nontrivial topological properties (point and line singularities in liquid crystals and ^{3}He, associated with the homotopy grounds π_1 and π_2 of various manifolds; the Polyakov–'t Hooft monopole, instantons, and various kinds of topologically nontrivial solutions). The methods of topology and algebraic geometry are known to play an important role in the theory of periodic solutions of equations of the Korteweg–deVries type (KdV) in the inverse problem method.

This study is a sequel to the survey by V. G. Drinfeld, I. M. Krichever, Y. I. Manin and S. P. Novikov, "*Methods of Algebraic Geometry in Contemporary Mathematical Physics*", first published in Soviet Science Reviews section C, by Harwood Academic Publishers (1980) and republished by Cambridge Scientific Publishers in 2004. The present survey can be regarded as its continuation, containing further material on this fundamental theme.

Classic Reviews in Mathematics and Mathematical Physics

A series of books on topics in mathematics and mathematical physics

Edited by S. P. Novikov and I. M. Krichever, L. D. Landau Institute of Theoretical Physics, Moscow, Russia

Volume 1
Triangle Equations and Simple Lie Algebras
A. A. Belavin and V. G. Drinfeld

Volume 2
Topological and Algebraic Geometry in Contemporary Mathematical Physics
B. A. Dubrovin, I. M. Krichever and S. P. Novikov

This book is part of a series. The publisher will accept continuation orders which provide for automatic billing and shipping of each title in the series upon publication. Please write for details.

Preface to the Series

One of the main motivation for publishing the series Soviet Scientific Reviews: Mathematical Physics Reviews in 1980 was to contribute to the development of scientific co-operation and better understanding among scientists by overcoming the language, communication and distribution difficulties. The review papers in this series were written by the very best Soviet experts and are now acknowledged as classic papers in particular areas of mathematics and mathematical physics. Written as scientific front line reviews, many of them could be used as reference books for the modern generation of students and young researchers. The lack of corresponding literature and ever and ever growing interest in theoretical and mathematical physics and the remarkable results of recent years in solition theory, the theory of quantum topological models and their applications in topology, algebraic and differential geometey are the main reasons for publishing the updated and annotated editions of these classic papers in a new series entitled Classic Reviews in Mathematics and Mathematical Physics.

We hope that the series will be a valuable addition to the literature already available. By making each paper available as a separate publication and changing distribution policy we hope to address a wider audience and see at least some of the volumes not only on library shelves but at work, on desks of our readers.

S. P. Novikov
I. M. Krichever

Contents

Preface to the Series

Chapter 1 **Hamiltonian Formalism and Variational-Topological Methods for Finding Periodic Trajectories of Conservative Dynamical Systems.** ***S. P. Novikov***

1.0 Introduction
1.1 Hamiltonion Formalism. Simplest Examples. Systems of Kirchoff Type.
1.2 Hamiltonian Formalism for Systems of Hydrodynamic Origin
1.3 What is Morse Theory
1.4 Equations of Kirchhoff Type and the Dirac Monopole
1.5 Multivalued Functionals and the Analog of the Morse Theory. The Periodic Problem for Equations of Kirchhoff Type. Chiral Fields in an External Field.

Chapter 2 **Algebraic Geometry Methods in the Theory of the Baxter–Yang Equations.** ***I. M. Krichever***

2.0 Introduction
2.1 "Vacuum Vectors and Algebraic Curves
2.2 The Baxter Equations
2.3 Solutions of Rank 2.

Chapter 3 **Multidimensional Theta Functions and their Application to the Integration of Nonlinear Equations.** ***B. A. Dubrovin***

3.0 Introduction
3.1 Information from the Theory of Theta Functions and Riemann Surfaces
3.2 Clebsch–Gordon–Baker–Akhiezer Functions. Integration of Nonlinear Equations
3.3 Application of Nonlinear Equations to Classical Problems of the Theory of Riemann Surfaces and their Theta Functions

Chapter 1

HAMILTONIAN FORMALISM AND VARIATIONAL-TOPOLOGICAL METHODS FOR FINDING PERIODIC TRAJECTORIES OF CONSERVATIVE DYNAMICAL SYSTEMS

S. P. NOVIKOV

1.0 Introduction

Dynamical systems describing real physical processes are always (or, "as a rule") Hamiltonian in one sense or another, if we can neglect energy dissipation. Not infrequently the Hamiltonian formalism has no "obvious" origin from a Lagrangian formalism as the result of a Legendre transformation and may not even admit global canonical coordinates. This applies in particular to many systems of hydrodynamical origin. In what follows (cf. Sections 1.1 and 1.2) we shall discuss various aspects of the Hamiltonian formalism in more detail. This is one of the purposes of this part of the survey. The other main purpose of this part is the description of topological methods for studying periodic trajectories. The point is that the overwhelming majority of nontrivial conservative systems with even two degrees of freedom are nonintegrable. As a result a complete investigation is made extremely difficult. After stationary points, periodic solutions are the simplest objects of the qualitative theory of dynamical systems; nevertheless even the solution of the problem of the existence of periodic trajectories is often highly nontrivial and requires the use of topological methods. The most developed and widely used method of this type is the theory of Morse and Lyusternik–Shnirelman (LShM), which combines the calculus of variations with the topology of functional spaces consisting of closed contours (curves) in the configuration space under study (cf. Section 1.3).

The use of the LShM theory requires, however, the use of a strictly positive definite Lagrangian formalism. It is thus already clear that in more general Hamiltonian systems, that have no Lagrangian origin, this theory is not generally applicable. Variational principles on phase trajectories never lead to positive definite functionals. Some extremely interesting systems—which we call systems of Kirchhoff type— curiously enough reduce to a problem that is mathematically isomorphic to the theory of a charged particle in the magnetic field of a "Dirac monopole" (cf. Section 1.4). Among systems of Kirchhoff type are, for example:

(a) the Kirchhoff equations of motion of a rigid body in an ideal compressible fluid (without vortices), at rest at infinity;
(b) the equation of motion of a rigid body with a fixed point in an axially symmetric force field;
(c) the Leggett equation for the spin dynamics in the low-temperature A and B phases of ^{3}He (nuclear magnetic resonance).

In such systems the equations of motion can in the last analysis be reduced to the principle of extremal action S. However, the action S is, from the global point of view a "multivalued" functional in the space of closed contours (smooth curves) on the sphere S^2, which after reduction serves as the configuration space. This means that δS is a single-valued quantity (1-form or convector) on the space of contours, but the "integrals over cycles" in the contour space of the quantity δS are nontrivial. Thus S is a multivalued functional (on the circle, for example, $d\varphi$ is a single-valued 1-form, but φ is multivalued function).

One of the purposes of Section 1.5 is to extend the topological methods of the LShM theory to multivalued functionals. This enables us to establish the existence of a large number of periodic orbits for systems of Kirchhoff type (cf. Section 1.5). Essentially we here present results of the papers [1], [2], [3].

1.1 Hamiltonian Formalism. Simplest Examples. Systems of Kirchhoff Type

From the modern point of view. "*Poisson brackets*" are the basis of the Hamiltonian formalism. Let y^i be coordinates on a manifold ("*phase space*") and $f(y), g(h)$ two function; the Poisson bracket is given by the tensor field $h^{ij}(y)$

$$\{f, g\} = h^{ij}(y)\frac{\partial f}{\partial y^i}\frac{\partial g}{\partial y^j}. \tag{1}$$

The following properties must be satisfied:

a) bilinearity and skew symmetry

$$\{f, g\} = -\{g, f\}; \tag{2$'$}$$

b) the Leibnitz identity

$$\{fg, h\} = f\{g, h\} + g\{f, h\}; \tag{2$''$}$$

c) the Jacobi identity

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0. \tag{2$'''$}$$

By definition, Hamiltonian systems have the form

$$\dot{f} = \{f, H\} \tag{3}$$

where f is any function and H is the Hamiltonian. There may be nontrivial (perhaps assigned locally on the manifold) functions f_j such that

$$\{f_j, g\} = 0 \tag{4}$$

for any function $g(y)$. In the case the Poisson bracket is said to be "degenerate": the matrix $h^{ij}(y)$ is degenerate. If we have found all such quantities $f_l(y)$, then on their common level surface

$$f_l(y) = \text{const} \tag{5}$$

the Poisson bracket becomes nondegenerate.

Let z^q be the coordinates on the level surface (5). The restriction of the tensor $h^{ql}(z)$ to this surface is nondegenerate, and has the inverse matrix

$$h_{ql}h^{lt} = \delta^t_q. \tag{6}$$

The inverse matrix determines a 2-form

$$\Omega = h_{q1} dz^q \wedge dz^l. \tag{7}$$

From (2) it follows that the form Ω is closed:

$$d\Omega = 0 \Leftrightarrow \frac{\partial h_{ql}}{\partial z^t} + \frac{\partial h_{tq}}{\partial z^l} + \frac{\partial h_{lt}}{\partial z^q} = 0. \tag{8}$$

Let us consider the main types of phase spaces.

Type I   Classical Hamiltonian formalism and variational principles

Let $(y) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$, the matrix h^{ij} is constant and nondegenerate:

$$h^{ij} = h_{ij} = \begin{pmatrix} 0 & & 1 & & 0 \\ & & & \ddots & \\ & & 0 & & 1 \\ -1 & & 0 & & \\ & \ddots & & & \\ 0 & & -1 & & 0 \end{pmatrix} = \text{const.} \tag{8$'$}$$

Equation (4) has the form:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}.$$

The coordinates (x, p) are said to be canonical. They can always be introduced locally for nondegenerate Poisson brackets (Darboux's theorem).

If $H(x, p)$ is the Hamiltonian, then we have a Lagrangian $L(\dot{x}, x)$, where x is the configuration space, determined from the equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad L = p_i \dot{x}^i - H. \tag{8$''$}$$

It is assumed that the equation $\dot{x}^i = \partial H / \partial p_i$ can be solved for the variables p_j. The Hamiltonian equations (4) are obtained from the variational principle $\partial S = 0$, where:

$$S = \int L(x, \dot{x}) dt. \tag{9}$$

Type II Hamiltonian formalism and Lie algebras

Now let us consider the next more complicated case, when the tensor h^{ij} is not constant, but depends linearly on the coordinates (y)

$$h^{ij} = C_k^{ij} y^k, \quad C_k^{ij} = \text{const.} \tag{10}$$

We consider the set L of all linear functions on the phase space, which we denote by L^*. For the basis linear forms, the coordinates y^i, there is defined an operation of "commutation" ($y^1 \in L^{**} = L$),

$$[y^i, y^j] = C_k^{ij} y^k = \{y^i, y^j\}. \tag{11}$$

From the requirements ($2', 2'', 2'''$) it follows that the operation (11) converts the linear space L into a Lie algebra, where the adjoint space L^* is the phase space for the Poisson bracket (10).

Examples

1. The basic example of a Hamiltonian formalism of type I is the phase space $T^*(M)$, the space of covectors (with lower indices) on the manifold M (configuration space). The manifold M may even be infinite-dimensional (space of fields $q(x)$ of any type, where x is one of the "indices" in the formulas). In the finite-dimensional case we have local coordinates x^i and conjugate momenta p_i, with Poisson brackets

$$\{x^i, x^j\} = \{p_i, p_j\} = 0, \quad \{x^i, p_j\} = \delta^i_j \tag{12}$$

and the form

$$\Omega_0 = \sum dx^i \wedge dp_i.$$

In the infinite-dimensional case we have fields and Poisson brackets of the form

$$\begin{aligned} \{q^i(x), p_j(y)\} &= \delta^i_j \delta(x-y) \\ \{q^i(x), q^j(y)\} &= \{p_i(x), p_j(y)\} = 0. \end{aligned} \tag{13}$$

2. It is important to consider also Poisson brackets of the form (12), distorted in addition by an "external field" $F_{ij}(x)$,

$$\{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = F_{ij}(x) \tag{14}$$

where the 2-form $F = F_{ij}dx^i \wedge dx^j$ is closed,

$$dF = 0.$$

We get a 2-form

$$\Omega = \sum dx^i \wedge dp_i + \sum F_{ij}dx^i \wedge dx^j = \Omega_0 + F. \qquad (15)$$

The equations of motion with the Hamiltonian $H(x, p)$ and the Poisson bracket (14) represent (for $n = 2, 3$) the equations of motion of a charged particle in an external magnetic field F_{ij} (or an electromagnetic field, for $n = 4$).

3. Somewhat more general, a priori, but as a rule reducible to the form (14), are Poisson brackets on the space $T^*(M)$, satisfying the requirement: any pair of functions (f, g) on the base M (independent of the variables p_i on the fibre, consisting of all covectors with a lower index) have a null Poisson bracket

$$\{f, g\} = 0. \qquad (16)$$

We call the property (16) "variational admissibility" of the Poisson bracket on $T^*(M)$. Obviously the bracket (14) is variationally admissible. As we know, over sufficiently small regions any (nondegenerate) Poisson bracket reduces to the form (12). Globally this is no longer so: if the form Ω is not exact, then the Poisson bracket does not reduce to the form (12). Variationally admissible Poisson brackets probably always reduce globally of the form (14), though this has not been rigorously shown; they reduce to the simplest form (12) over any region where the form Ω is exact.

We now proceed to discuss examples of Poisson brackets of type **II**, associated with Lie algebras.

1. Suppose that L is the Lie algebra of the group SO_3. The Killing metric is Euclidean, so we need not distinguish between L and L^*. The Poisson bracket of basis functions M_i on L^* has the form (11),

$$\{M_i, M_j\} = \epsilon_{ijk}M_k, \quad C_k^{ij} \to \epsilon_{ijk}. \qquad (17)$$

There is a function $M^2 = \sum M_i^2$, such that

$$\{M^2, M_i\} = 0, \quad i = 1, 2, 3. \qquad (18)$$

Hamiltonian systems have the form

$$\dot{M}_i = \{M_i, H(M)\}. \tag{19}$$

Suppose that $\Omega^i = \partial H/\partial M_i$; the Killing metric permits us to make no distinction between upper and lower indices. The equation (19) reduce to the form of the "Euler equation"

$$\dot{M}_i = [M, \Omega]. \tag{20}$$

This conclusion is valid for all compact Lie groups in which there is a Killing metric, a euclidean metric on the Lie algebra, invariant under all inner automorphisms

$$L \to gLg^{-1} \tag{21}$$

where g is an element of the Lie group, and L is its Lie algebra. We recall that the Poisson bracket (11) is invariant only with respect to the transformations (21). For the classical Euler equations of free rotation of a rigid body we have

$$G = SO_3, \quad H = \sum a_i M_i^2/2. \tag{22}$$

Ω is the angular velocity of the body, and M is its angular momentum.

2. Many important systems that arise in hydrodynamics are associated with the Lie algebra of the group $E(3)$ of motions in the euclidean space R^3. This algebra is no longer *semisimple*. In the phase space L^* there are 6 coordinates $(M_1, M_2, M_3, p_1, p_2, p_3)$ and the Poisson brackets are

$$\{M_i, M_j\} = \epsilon_{ijk} M_k, \quad \{M_i, p_j\} = \epsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0. \tag{23}$$

The bracket (23) has a pair of independent functions:

$$f_1 = \sum p_i^2 = p^2, \quad f_2 = \sum M_i p_i = ps,$$

such that

$$\{f_q, M_i\} = \{f_q, p_i\} = 0, \quad q = 1, 2. \tag{24}$$

Let $H(M, p)$ be the Hamiltonian. We introduce the notation $u' = \partial H/\partial p_i$, $\omega^i = \partial H/\partial M_i$. The Hamiltonian equations take the "Kirchhoff form":

$$\dot{p} = [p \times \omega], \quad \dot{M} = [M \times \omega] + [p \times u]. \tag{25}$$

Equations (25) coincide (for quadratic Hamiltonians H) with the Kirchhoff equations for the motion of a rigid body in a liquid that is ideal, incompressible, and at rest at infinity [4]. The motion of the liquid itself is assumed to be potential. In this case, H is the energy, and M and p are the total angular momentum and linear momentum of the body-liquid system in the movable coordinate system rigidly attached to the body. The energy H is assumed to be positive and quadratic in both variables M and p. Using the transformations (21) the form H can be changed to

$$2H = \sum a_i M_i^2 + \sum b_{ij}(M_i p_j + p_i M_j) + \sum c_{ij} p_i p_j. \tag{26}$$

Let us consider two other applications of Eqs. (25):

A) The equations of motion of a rigid body with a fixed point in an axisymmetric force field with potential $W(z)$ take the form (25). The corresponding Hamiltonian has the form

$$H = \sum a_i M_i^2/2 + W(l^i p_i) \tag{27}$$

where l^i is a vector determined by the position of the center of mass relative to the inertial axis and the fixed point. The quantities p_i here are dimensionless and do not have the physical significance of momenta. They are the direction cosines of some unit vector, i.e.,

$$f_1 = \sum p_i^2 = 1. \tag{27'}$$

B) The (Leggett) equation for the dynamics of the spin in the A-phase of superfluid ^{3}He also reduces to the form (25); this is the dynamics of the spin-variables, the vectors (s, d), where $d^2 = 1$ in analogy to (27). (Cf. the survey of Brinkman and Cross in the book [5].) To get the Leggett equations for nuclear magnetic resonance in the A-phase we must make the changes in notation

$$M_i \to s_i, \quad p_i \to d_i$$

(where s is the "magnitic moment"), and consider a Hamiltonian of the form

$$H = \frac{1}{2} a s^2 + b(s_i d_i)^2 + \lambda(s_i H_i) + W(d). \tag{28}$$

Here a, b, λ are constants, H^i is the external magnetic field, and the potential W has the form:

$$W(d) = \text{const} \cdot (l^i d_i)^2. \tag{28'}$$

Because of property (24) of the Poisson bracket (23), the quantity $s_i d_i = f_2$ is equivalent to a constant in the equations of motion. Therefore the second term in the Hamiltonian can simply be omitted:

$$H \sim H' = \frac{1}{2} a s^2 + \lambda s_i H_i + W(d). \tag{28''}$$

The d-spin part of the so-called "order parameter" is a unit vector $d^2 = 1$, as already pointed out earlier.

3. There is also another phase, the B-phase of ^{3}He, in which the Leggett equation takes a form different from the classical top (cf., for example, the survey of Brinkman and Cross in the book [5]).

In the state of hydrodynamic rest and with nonzero spin, the state in the B-phase is determined by the pair: of the rotation matrix $R = (R_{ij}) \in SO_3$ and (s_i), $i = 1, 2, 3$, the "magnetic moment."

The variables s_i are coordinates in the adjoint space to the Lie algebra of the group SO_3, analogous to the angular momenta M_i. The standard Poisson bracket on $T^*(SO_3)$ in the variables $(S_{i, R_{jk}})$ is written as

$$\{s_i, s_j\} = \epsilon_{ijk} s_k, \quad \{R_{ij}, R_{kl}\} = 0, \quad \{s_i, R_{jl}\} = \epsilon_{ijk} R_{kl}. \tag{29}$$

The Hamiltonian of the Leggett system in the B-phase in an external magnetic field has the form

$$H = \frac{1}{2} a s^2 + b s_i F_i + V(\cos\theta) \tag{30}$$

where a and b are constants, $F = (F_i)$ is the external field, and

$$V(\cos\theta) = \text{const} \cdot \left(\frac{1}{2} + 2\cos\theta\right)^2. \tag{31}$$

R_{ij} is the rotation through angle θ around the axis n_i, $n^2 = 1$:

$$\begin{aligned} R_{ij} &= \cos\theta \delta_{ij} + (1 - \cos\theta) n_i n_j + \sin\theta \epsilon_{ijk} n_k \\ 1 + 2\cos\theta &= R_{ii} = \text{Sp}(R_{ij}). \end{aligned} \tag{32}$$

After the replacements

$$aS_i = \omega_i, \quad \Omega_{jk} = \epsilon_{jki}\omega_i = (\dot{R}R^{-1})jk \tag{33}$$

we obtain a Lagrangian system in the variables $(\dot{R}_{ij}, R_{ij})$ on $T^*(SO_3)$, where the kinetic energy is determined by the two-sided invariant Killing metric, and the potential energy $V(\cos\theta)$ is invariant under inner automorphisms

$$R \to gRg^{-1}, \quad s \to gs \quad g \in SO_3. \tag{34}$$

If the field $F = (F_i)$ is constant, then the whole Lagrangian is invariant under the one-parameter group of transformations (34), where g belongs to the group of rotations around the axis of the field F. Let us assume that $F = (F, 0, 0)$.

For zero field $F = 0$, the system admits the group SO_3 of transformations (34) and was completely integrated in [6]. The transformations (34) generate a conserved vector ($F = 0$)

$$A_j = (1 - \cos\theta)\left[n \times \left(\cot\frac{\theta}{2}s + [n \times s]\right)\right] \tag{35}$$

where, as for the ordinary angular momenta, the Poisson brackets are

$$\{A_i, A_j\} = \epsilon_{ijk}A_k, \quad \left\{A_i, \frac{1}{2}as^2 + V(\cos\theta)\right\} = 0. \tag{36}$$

As pointed out by V. L. Golo [7], the variables s^2 and θ, appearing in the Hamiltonian when $F = 0$, generate a closed algebra of Poisson brackets $\{s^2, s_\parallel, \theta\}$, where

$$\begin{gathered} s_\parallel = s_i n_i, \quad \{s^2, \theta\} = 2s_\parallel \\ \{s_\parallel, \theta\} = 1, \quad \{s^s, s_\parallel\} = \frac{1 + \cos\theta}{\sin\theta}(s^2 - s_\parallel^2). \end{gathered} \tag{37}$$

The quantity $A^2 = \sum A_i^2 = (1 - \cos\theta)(s^2 - s_\parallel^2)$ has null Poisson bracket with this whole subalgebra

$$\{A^2, s^2\} = \{A^2, s_\parallel\} = \{A^2, \theta\} = 0. \tag{38}$$

In a nonzero field $(F, 0, 0)$ there remains only one integral in addition to the energy*

$$\{A_1, H\} = 0. \tag{39}$$

*For large fields, $F \to \infty$, this system, including viscosity, was investigated in [8].