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AN ANALYSIS OF EXPLICIT FINITE ELEMENT APPROXIMATIONS FOR THE SHALLOW WATER EQUATIONS

by

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ABSTRACT

Schemes for generating stable explicit algorithms for transient analysis of the shallow water equations are analysed. Particular emphasis is given to numerical dissipation and spatial as well as time stability. Examples include a one-dimensional travelling wave, tidal flow and shoaling.

INTRODUCTION

In this paper we consider the application of the finite element method to environmental flow problems governed by the shallow water equations. From the computational point of view, the use of an explicit finite difference in time has advantages of low core storage requirement and simple treatment of nonlinearity for transient problems such as storm surge and Tsunami analysis. Numerical difficulties however arise from the hyperbolic nature of the equations and the nonlinear convective acceleration terms which make the standard Euler forward difference in time unconditionally unstable for consistent or lumped mass.

A special local approximation of the mass matrix has been introduced by one of the authors to obtain a stable explicit scheme [1]. This scheme has been successfully applied to practical problems [2,3]. Also in recent years upwind finite element approximations have been developed for second order elliptic problems with significant first derivatives [4] and their application to first order hyperbolics has been suggested in [5]. Here we apply the Petrov-Galerkin finite element approximations to the shallow water wave equations. In a simple one-dimensional problem with linear interpolation and mass lumping the effect of upwind weighting is identified as artifical dissipation. Generalising this result we damp the Galerkin approximations to obtain a formulation which is easily extended to two-dimensional and nonlinear problems. This approach also avoids the directional property of upwinding and can easily be implemented in transient problems.

The selective mass lumping and damped Galerkin schemes are described in the following sections. Special attention is given to their dissipative properties and both time and spatial stability. The performance of the schemes is then tested on some simple examples.

PRELIMINARIES

Defining $u_j (j=1,2)$ to be velocity components in $x_j (j=1,2)$ directions and η the surface elevation, the vertically integrated Navier-Stokes equations for shallow water are given by

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + g \frac{\partial \eta}{\partial x_i} = \mu \frac{\partial}{\partial x_i} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + f_i \tag{1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i} (H + \eta) u_i = 0. \tag{2}$$

The summation convention on repeated indices is used, and H, g, μ and $f_j (j=1,2)$ denote the depth, gravity, viscosity of the fluid and nonlinear forces such as bed friction. Omitting the nonlinear and viscous terms from these equations gives the familiar long wave equations in one dimension.

$$\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} = 0$$

$$x \in (0,1), \quad t \in (0,T]$$

$$\frac{\partial \eta}{\partial t} + H \frac{\partial u}{\partial x} = 0 \tag{3}$$

Applying the transformation

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \sqrt{H} & -\sqrt{g} \\ \sqrt{H} & \sqrt{g} \end{bmatrix} \begin{bmatrix} u \\ \eta \end{bmatrix} \tag{4}$$

to Eqs.3 symmetrices these equations so that the system becomes

$$\frac{\partial v_i}{\partial t} + c \frac{\partial v_i}{\partial x} = 0 \qquad i = 1,2 \tag{5}$$

where

$$c = (-1)^i \sqrt{gH} .$$

EXPLICIT FINITE ELEMENT APPROXIMATIONS

For simplicity the present analysis will be applied to the simple transport equation (5). Considerations of stability and dissipation are directly applicable to the one-dimensional long wave equation but must be generalised for the full two-dimensional nonlinear shallow water equations. The standard Galerkin approximation with linear trial functions of Eq. 5 with an Euler forward difference in time gives

$$\underset{\sim}{M} \frac{\underset{\sim}{v}^{n+1} - \underset{\sim}{v}^{n}}{\Delta t} + \underset{\sim}{K} \underset{\sim}{v}^n = 0 \tag{6}$$

or recursively

$$\underset{\sim}{M} \underset{\sim}{v}^{n+1} = \underset{\sim}{M} \underset{\sim}{v}^{n} - \Delta t \underset{\sim}{K} \underset{\sim}{v}^{n} . \tag{7}$$

Here all the coefficient matrices have the same tridiagonal banded structure with the band terms given by

$$\underset{\sim}{M}_i = \frac{h}{6} \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}$$

and

$$\underset{\sim}{K}_i = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} .$$

The lumped mass approximation $\bar{\underset{\sim}{M}}$ is given as

$$\bar{\underset{\sim}{M}} = h \left\lceil 1 \right\rfloor \quad .$$

A lumped mass approximation to the left hand side only of Eq. (7) was introduced in [1] and used with two-step Lax Wendroff time marching schemes. Details of the stability and convergence are given in [6]. Applying this selective lumping scheme to Eq. 7 we obtain the following difference expression for the i-th node at the n-th time step.

$$h \; v_i^{n+1} = \frac{h}{6} (v_{i-1}^n + 4v_i^n + v_{i+1}^n) - \frac{c\Delta t}{2} (v_{i+1}^n - v_{i-1}^n) \tag{8}$$

Using Taylor series expansions of the solution at t=nΔt and x=ih we can identify the truncation errors in Eq. 8 by writing

$$\frac{\partial v_i^n}{\partial t} + c \frac{\partial v_i^n}{\partial x} = \frac{h^2}{6\Delta t} \frac{\partial^2 v_i^n}{\partial x^2} - \frac{\Delta t}{2} \frac{\partial^2 v_i^n}{\partial t^2} - \frac{ch^2}{6} \frac{\partial^3 v_i^n}{\partial x^3} - \frac{\Delta t^2}{6} \frac{\partial^3 v_i^n}{\partial t^3} + O(h^3) \quad . \tag{9}$$

Differentiation of Eq. 5 first with respect to t and then with respect to x and subtracting identifies

$$\frac{\partial^2 v}{\partial t^2} = - c^2 \frac{\partial^2 v}{\partial x^2}$$

Substituting in (9)

$$\frac{\partial v_i^n}{\partial t} + c \frac{\partial v_i^n}{\partial x} = \left(\frac{h^2}{6\Delta t} - \frac{c^2 \Delta t}{2}\right) \frac{\partial^2 v_i^n}{\partial x^2} + \frac{c}{6} (c^2 \Delta t^2 - h^2) \frac{\partial^3 v_i^n}{\partial x^3} + O(h^3) \tag{10}$$

Comparing Eq. 10 and Eq. 5 we can see that the Euler forward difference has introduced a negative diffusion proportional to Δt which normally leads to numerical instability. The selective lumping of the mass matrix gives a positive diffusion with coefficient $h^2/6\Delta t$. Obviously the critical time step for this scheme

$$\Delta t_{crit} = \frac{h}{\sqrt{3}c} \quad . \tag{11}$$

An alternative explicit scheme can be obtained using the "upwind" finite elements described in [4]. Applying the Petrov-Galerkin method the weak form of Eq. 5 is given by

$$(w_h^*, \frac{\partial v_h}{\partial t} + c \frac{\partial v_h}{\partial x}) = 0$$

Applying the "upwind" test functions

$$w_h^* = \begin{cases} N_j - \alpha\left[\frac{3}{h^2}\, x\,(x-h)\right] & x\varepsilon((j-1)h, jh) \\ N_j + \alpha\left[\frac{3}{h^2}\,(x-h)(x-2h)\right] & x\varepsilon(jh,(j+1)h) \end{cases}$$

and the normal linear trial functions in space, the difference expression corresponding to Eq. (8) with lumped mass matrix is

$$h\, v_i^{n+1} = hv_i^n - \frac{c\Delta t}{2}\left[(1+\alpha)v_{i+1}^n - 2\alpha v_i^n - (1-\alpha)v_{i-1}^n\right]. \qquad (12)$$

Again using Taylor series expansions we can identify the added dissipation by writing

$$\frac{\partial v_i^n}{\partial t} + c\,\frac{\partial v_i^n}{\partial x} = \left(\frac{\alpha ch}{2} - \frac{c^2\Delta t}{2}\right)\frac{\partial^2 v_i^n}{\partial x^2} + \frac{c}{6}\,(c^2h^2 - \Delta t^2)\,\frac{\partial^3 v_i^n}{\partial x} + O\,(h^3)\,. \qquad (13)$$

Note that if we choose α to cancel the numerical diffusion as

$$\alpha = \frac{c\Delta t}{h}$$

the resulting recursion formula is exactly the same as the Lax-Wendroff finite difference scheme.

Difficulties arise when extending the upwind model to two-dimensions. These difficulties can however be avoided if we generalise the diffusive characteristic of the Petrov-Galerkin scheme by defining the weak form of Eq. 5 as

$$\left(w_h, \frac{\partial v_h}{\partial t} + c\,\frac{\partial v_h}{\partial x}\right) + \gamma\left(\frac{\partial w_h}{\partial x}, \frac{\partial v_h}{\partial x}\right) = 0 \qquad (14)$$

where γ is a coefficient of artificial viscosity and has the property that $\gamma\to 0$ as $h\to 0$. A similar scheme has been proposed in [7] with Hermite cubics. The use of artificial viscosity in continuous-in-time Galerkin approximations has also been shown to improve the rate of convergence of the standard Galerkin procedure [8]. In [9] the Petrov-Galerkin method for convective-diffusion is interpretted in a similar manner.

The lumped mass discretized form of the damped Galerkin scheme defined by Eq. 14 has the difference expression

$$hv_j^{n+1} = hv_j^n - \frac{c\Delta t}{2}\,(v_{i+1}^n - v_{i-1}^n) + \gamma\frac{\Delta t}{h}\,(v_{i+1}^n - 2v_i^n + v_{i-1}^n) \qquad (15)$$

and truncation errors are again identified by writing

$$\frac{\partial v_i^n}{\partial t} + c\,\frac{\partial v_i^n}{\partial x} = \left(\gamma - \frac{c^2\Delta t}{2}\right)\frac{\partial^2 v_i^n}{\partial x^2} + \frac{c}{6}\,(c^2\Delta t^2 - h^2)\,\frac{\partial^3 v_i^n}{\partial x^3} + O\,(h^3)\,. \qquad (16)$$

Choosing $\gamma=c^2\Delta t/2$ we again obtain the Lax-Wendroff finite difference scheme which corresponds to the minimum requirement of numerical diffusion. Also choosing $\gamma=\alpha ch/2$ we reproduce the Petrov-Galerkin upwind scheme, and $\gamma=h^2/6\Delta t$ the selectively lumped scheme.

It is interesting to note that the scheme defined by Eq. 15 can realize the unit CFL (Courant, Friedrichs-Lewy) property that the wave propagates from

v_i^n to v_{i+1}^{n+1} exactly when we choose $\Delta t = c/h$. All practical finite difference schemes have this property. The critical time step defined by Eq. 11 however excludes this property from the selectively lumped scheme.

STABILITY OF THE EXPLICIT SCHEMES

Following the procedure described in [3] we assume the solution at $t=n\Delta t$ on $x=ih$ to be of the form

$$v_i^n = y^n e^{j\omega ih} \tag{17}$$

where $j=\sqrt{-1}$. The condition for stable solution $\left| \frac{y^{n+1}}{y^n} \right| \le 1$ requires that

$$4\left(\frac{Cr}{Pe^*}\right) + 4\left(\frac{Cr}{Pe^*}\right)^2 (\cos \omega h - 1) - Cr^2 (\cos \omega h - 1) \ge 0 \tag{18}$$

with $Cr = \frac{c\Delta t}{h}$ and $Pe^* = \frac{ch}{\gamma}$.

From the existence of the solution, we have the condition for wave propagation

$$Cr \le 1 \tag{19}$$

For the range $Pe^* \le 2$ where the diffusive part dominates the solution we have

$$Pe^* \ge 2\ Cr \tag{20}$$

and for the range $Pe^* \ge 2$ we get a condition

$$Pe^* \le \frac{2}{Cr} \tag{21}$$

Note that the condition (21) corresponds to the minimum requirement of diffusion in Eq. 16. If we satisfy condition (20) with $Pe^* \le 2$ an oscillation free solution in both space and time will be obtained because $Pe^* \le 2$ defines the condition for freedom from spatial oscillation.

If condition (21) is satisfied the selective lumping scheme is stable because substitution of

$$\gamma = \frac{h^2}{6\Delta t}$$

identifies

$$Pe^* = 6\ Cr$$

for this scheme. This result however suggests that this approximation can produce an excessive amount of numerical diffusion.

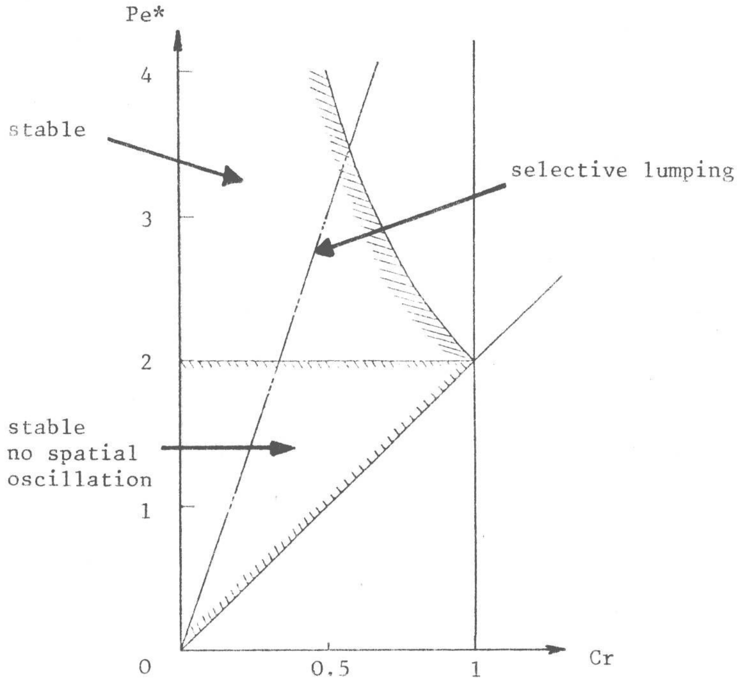


Figure 1 Stable region for the explicit schemes

NUMERICAL EXPERIMENTS ON A ONE-DIMENSIONAL TRAVELLING WAVE

To assess the explicit schemes we firstly look at the behaviour of the solution for a simple example of Eq.5 with initial condition

$$v_o = \begin{cases} 0 & 0.5 < x \leq 1 \\ 1 & 0 < x \leq 0.5 \end{cases}$$

with $c=1.0$ and $v(0,t)=v(1,t)$. Taking $h=0.1$ and $\Delta t=0.1$ we obtain the solution corresponding to the unit CFL property which is shown as the square pulse in Fig. 2. In practical computation however we use Δt less than the unit CFL value to achieve stability in irregular grids.

A series of numerical results are given in Fig. 2 for $\Delta t=0.05$ and 0.01 when the critical value for stability is $\Delta t=0.1$. The solution at $t=0.2$ is given in Fig. 2a and Fig. 2b for the damped Galerkin and selective lumping respectively. Since the problem is closed spatially and there is no energy leak through the boundary we have

$$\frac{\partial}{\partial t} || v ||_o = 0$$

where $|| \cdot ||_o$ is the usual L^2 norm [10]. As shown in Fig. 3, this property is better approximated by the damped Galerkin procedure than by the selective lumping scheme when appropriate control of the numerical damping is exercised.

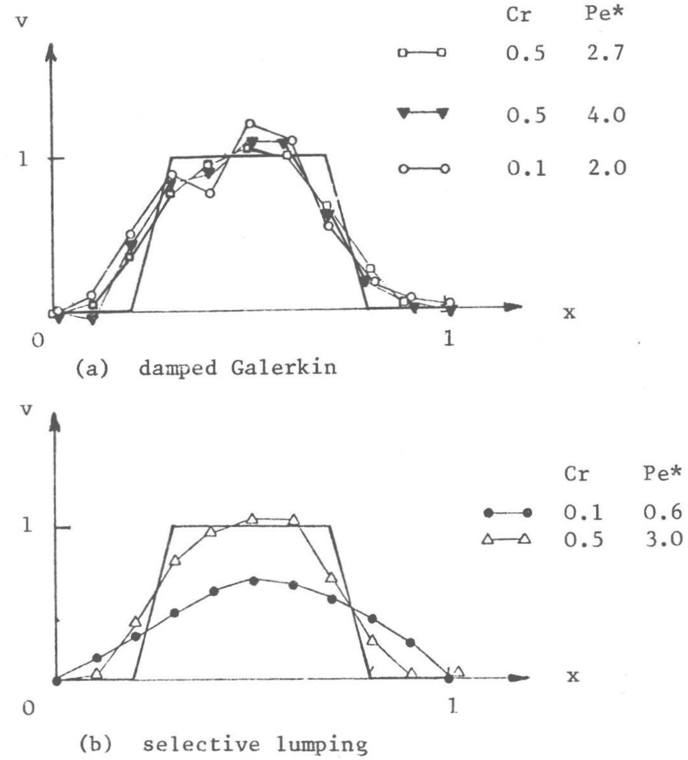


Figure 2 Solutions for one-dimensional travelling wave (c=1, h=0.1)

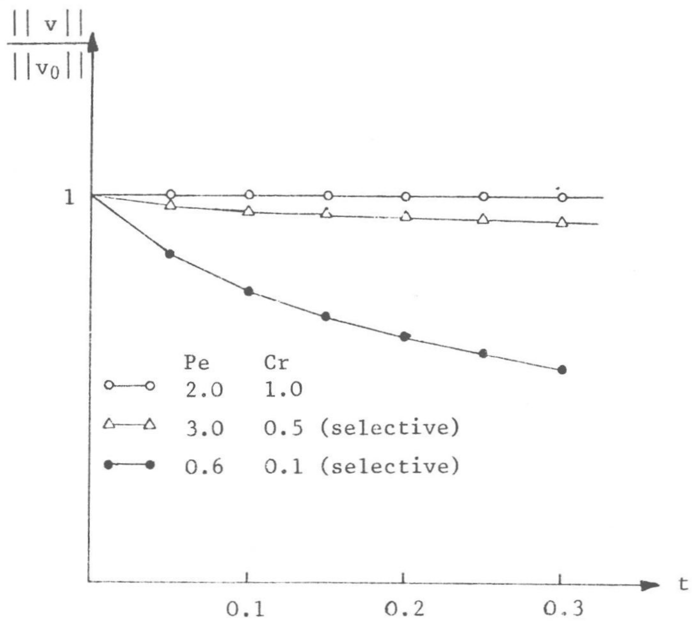


Figure 3 Conservative properties of the numerical schemes

To investigate these dissipative characteristics further we write Eq. 7 as

$$\underset{\sim}{v}^{n+1} = \underset{\sim}{A}\,\underset{\sim}{v}^{n}$$

where $\underset{\sim}{A}$ is the amplification matrix of the explicit schemes applied to the symmetrised equation (5). The complex eigenvalues of $\underset{\sim}{A}$ are the participation coefficients for the eigenmodes and in the absence of any diffusion should all have unit modulus. In Table 1 we give the modulus of the eigenvalues of $\underset{\sim}{A}$ for selective lumping and the damped Galerkin schemes applied to the one-dimensional region. Also given is the participation of the eigenmodes to give the initial conditions used above. The modes affected by the numerical dissipation can be seen to make a significant contribution to the solution. (Note that all eigenvalues and mode participation factors are complex and only their modulus is given here).

Selective lumping		Damped Galerkin			Modal participation for travelling wave (t = 0)
Cr = 0.1	Cr = 0.5	Cr = 0.1, Pe = 20	Cr = 0.5, Pe = 2.7	Cr = 1.0, Pe = 2.0	
1.0	1.0	1.0	1.0	1.0	1.508
0.949	0.985	1.0	0.979	1.0	1.059
0.949	0.985	1.0	0.979	1.0	1.059
0.810	0.925	0.998	0.906	1.0	0.157
0.810	0.925	0.998	0.906	1.0	0.157
0.627	0.793	0.994	0.760	1.0	0.363
0.627	0.793	0.994	0.760	1.0	0.363
0.454	0.586	0.986	0.541	1.0	0.179
0.454	0.586	0.986	0.541	1.0	0.179
0.348	0.374	0.981	0.294	1.0	0.230
0.348	0.374	0.981	0.294	1.0	0.230

Table 1. Modulus of eigenvalues of amplification matrix.

NUMERICAL EXPERIMENTS ON A ONE-DIMENSIONAL BORE

The one dimensional bore problem is shown in Fig. 4. We use h=0.025 and Δt=0.01 for a series of experiments where g and H are taken to be unity giving Cr=0.4. Fig. 4 shows the deformation of the wave at t=0.1 for different Peclet numbers. As predicted by the stability analysis, for $Pe \leq 2$ the solutions are stable in time and space. For $2 < Pe < \frac{2}{Cr}$ the solution is stable in time but spatial oscillations occur.

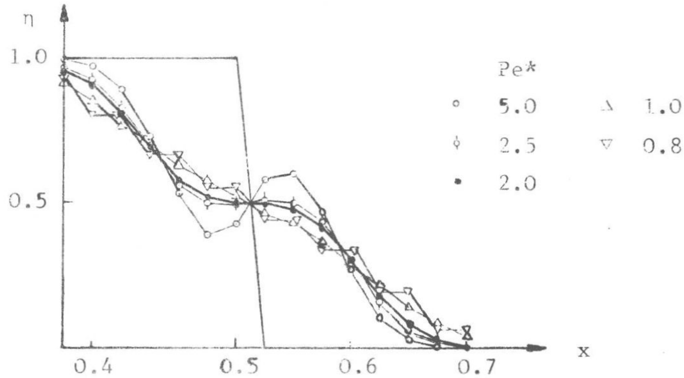


Figure 4 Deformation of wave front of one-dimensional bore

EXTENSION OF THE EXPLICIT SCHEMES TO THE NONLINEAR SHALLOW WATER EQUATIONS

The analysis of the previous sections has been applied to one of the symmetrised equations (5). Writing the difference expression (15) for both characteristics and applying the transformation (4) leads to the matrix equation

$$\begin{bmatrix} \dot{u}_i \\ \dot{\eta}_i \end{bmatrix}^n = \frac{1}{2h}\left[\begin{bmatrix} \frac{2\gamma}{h} & g \\ H & \frac{2\gamma}{h} \end{bmatrix} \begin{bmatrix} u_{i-1} \\ \eta_{i-1} \end{bmatrix}^n + \begin{bmatrix} \frac{-4\gamma}{h} & 0 \\ 0 & \frac{-4\gamma}{h} \end{bmatrix} \begin{bmatrix} u_i \\ \eta_i \end{bmatrix}^n + \begin{bmatrix} \frac{2\gamma}{h} & -g \\ -H & \frac{2\gamma}{h} \end{bmatrix} \begin{bmatrix} u_{i+1} \\ \eta_{i+1} \end{bmatrix}^n \right] \quad \ldots\ldots (22)$$

which can be identified as

$$\frac{\partial u_i^n}{\partial t} + g\,\frac{\partial \eta_i^n}{\partial x} = \gamma\,\frac{\partial^2 u_i^n}{\partial x^2}$$

and

$$\frac{\partial \eta_i^n}{\partial t} + H\,\frac{\partial u_i^n}{\partial x} = \gamma\,\frac{\partial^2 \eta_i^n}{\partial x^2} \qquad (23)$$

To extend this damped Galerkin approximation to two-dimensional problems including nonlinear acceleration consider triangulation of the bounded domain Ω and the weak form

$$(w,\frac{\partial u_i}{\partial t}) + (w, u_j\,\frac{\partial u_i}{\partial x_j}) + g\,(w,\frac{\partial \eta_i}{\partial x_j}) = 0 \qquad (24)$$

$$(w,\frac{\partial \eta_i}{\partial t}) + (w,\frac{\partial}{\partial x_j}(H+\eta)u_i) = 0 \qquad (25)$$

where the viscous term and body forces are neglected. On each triangle we construct a finite element approximation u_h, η_h and test functions w_h by using linear interpolation functions. Then the damped Galerkin finite element approximation is given by

$$(\bar{w}_h,\frac{\partial u_{ih}}{\partial t}) + (w_h, u_{jh}\,\frac{\partial u_{ih}}{\partial x}) + g\,(w_h,\frac{\partial \eta_{ih}}{\partial x_j}) + \gamma\,(\frac{\partial w_h}{\partial x_j}, \frac{\partial u_i}{\partial x_j}) = 0$$

$$(\bar{w}_h,\frac{\partial \eta_{ih}}{\partial t}) + (w_h,\frac{\partial}{\partial x_j}(H+\eta_{ih})u_{jh}) + \gamma(\frac{\partial w_h}{\partial x_j},\frac{\partial \eta}{\partial x_j}) = 0 \qquad (26)$$

By a Taylor series expansion at $t=n\Delta t$ we can determine the minimum requirement for numerical damping for the explicit time integration scheme to be [11]

$$\gamma_j = \left[\; g(H+\eta) + |u_j|^2 \;\right] \frac{\Delta t}{2} \qquad (27)$$

for the j-th component. We have taken the variables in Eq. 27 to be averages on the element.

At present the stability analysis summarized in Fig. 1 has been used as a qualitative guide for the two-dimensional nonlinear program with

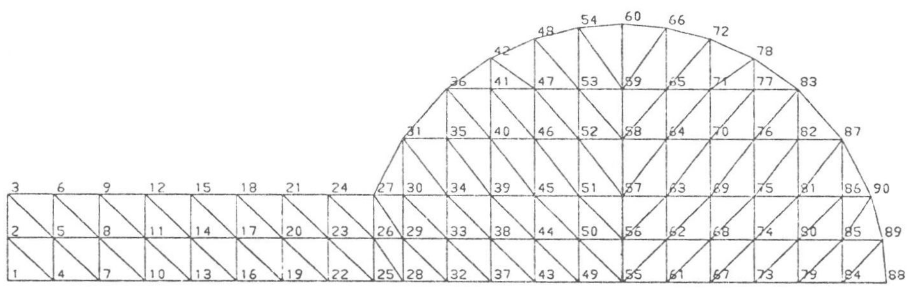
$$Cr^e = \frac{(\sqrt{g(H+\eta)} + u)\ \Delta t}{h}$$

and h the length of the smallest side on the elements. The examples show that stable schemes can be obtained.

TIDAL ELEVATION IN A CLOSED BASIN

A periodic twelve hour water elevation of 0.5 metre amplitude was applied to nodes 1, 2 and 3 of the constant 10 metre deep tidal basin shown in Fig. 5. The water surface elevation at node 55 at the centre of the basin is plotted in the figure. The effect of numerical diffusion is negligible in this example because the depth of the water in the basin is almost uniform at all times and there is little difference in the solution for the selective lumping and damped Galerkin schemes. The results are essentially identical to those reported in [2].

For this example a time step size of 30 seconds was taken so that the solutions required more than 4000 time steps. Both schemes however only required approximately 50 seconds CPU time on a CDC 7600 and ran in less than 20 K words of memory



Finite element discretization

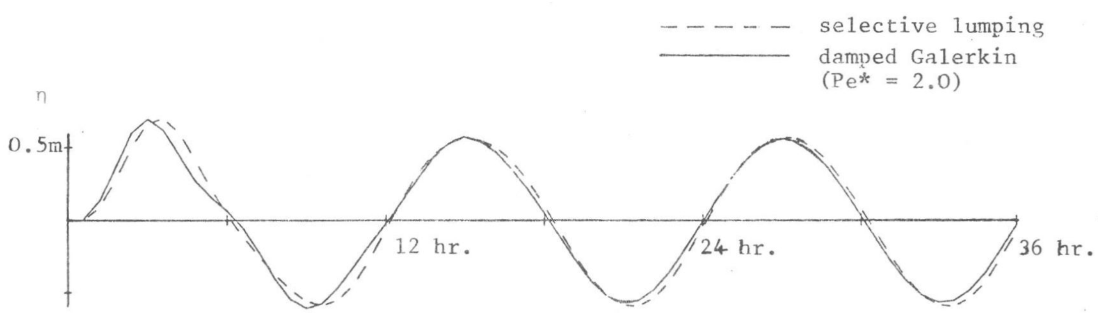


Figure 5 Tidal elevation in circular basin (Cr = 0.3)