Applied Partial Differential Equations

REVISED EDITION

$$a\left[\frac{\partial^2 u}{\partial x^2}\right]_-^+ + b\left[\frac{\partial^2 u}{\partial x \partial y}\right]_-^+ + \frac{\partial a}{\partial x}\left[\frac{\partial u}{\partial x}\right]_-^+ + \frac{\partial b}{\partial x}\left[\frac{\partial u}{\partial y}\right]_-^+ = \alpha\left[\frac{\partial u}{\partial x}\right]_-^+$$

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Applied Partial Differential Equations

Preface to the revised edition

The revision of this book has been a source of both pleasure and pain to the authors. The pleasure has come from the opportunity to include new material, almost all of which is intended to unify and tie together further the existing topics, and to give the reader the best possible overview of the wonderful interplay between partial differential equations and their real-world applications. This is all in keeping with our unshakeable philosophy that partial differential equations offer fabulously effective data compression: the basically simple structure of many partial differential equations enables knowledge holders to make a quantitative model of almost any 'continuous' process going on around them.

The pain of revision has come from realising that on many occasions our zeal in writing the first edition overstretched our accuracy. However, we have made amends as scrupulously as we can; we have been immensely helped in this task, and with the incorporation of new material, by the helpful comments of many of our colleagues and collaborators. We are also very grateful to Alison Jones and colleagues at Oxford University Press for their invaluable assistance in the final stages of publication.

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January 2003

Preface to the first edition

In the 1960s, Alan Tayler, Leslie Fox and their colleagues in Oxford initiated 'Study Group' workshops in which academic mathematicians and industrial researchers worked together on problems of practical significance. They were soon able to show the world that mathematics can provide invaluable insight for researchers in many industries, and not just those which at the time employed professional mathematicians.

This message is the theme of Alan's book *Mathematical methods in applied mechanics* [43], which contains many examples of how mathematical modelling and applied analysis can be put to work. That book revealed the ubiquity of partial differential equation models, but it did not lay out a co-ordinated account of the theory of these equations from an applied perspective. Hence this complementary volume was planned in the 1980s, first emerging as very informal lecture notes.

Much has happened since then. Alan's illness brought about two authorship changes: first, Andrew Lacey and Sasha Movchan stepped in to help, and, after Alan's tragic death in 1995, Sam Howison became involved as well. Additionally, the past decade has seen many new practical illustrations and theoretical advances which have been incorporated into the book, while still keeping it at around first-year graduate level.

Only now can we see the debt we owe not only to Alan Tayler but also to those who have supported us over the past ten years. In particular, we thank June Tayler, Annabel Ralphs, Natasha Movchan and Hilary Ockendon for their forbearance, Brenda Willoughby for typing help at a crucial stage, and Elizabeth Johnston and her colleagues at Oxford University Press.

A book like this cannot be written without help from colleagues around the world, far too many to mention here, but we would especially like to acknowledge the many helpful comments we have received from post-docs, who are often the most important people at the interface between mathematics and the real world.

Oxford J. R. O.
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February 1999

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Introduction

Partial differential equations are central to mathematics, whether pure or applied. They arise in mathematical models whose dependent variables vary continuously as functions of several independent variables, usually space and time. Their most striking attribute is their universality, a property which has enabled us to motivate every mathematical idea in this book by real-world examples from fluid or solid mechanics, electromagnetism, probability, finance and a host of other areas of application. Moreover, this applicability is growing day by day because of the flexibility and power of modern software tailored to suitable discretised approximations of the equations. Equally dramatic is the way in which the equations that arise in all these areas of application can so easily motivate the study of fundamental mathematical questions of great depth and significance and, conversely, benefit from the results of such investigations.

Whether or not it is in the context of a model of a physical situation, the analysis of a partial differential equation has many objectives. One of our principal goals will be to investigate the question of *well-posedness*. We will give a more precise definition of this in Chapter 2 but, roughly speaking, a partial differential equation problem is said to be well posed if it has a solution, that solution is unique, and it only changes by a small amount in response to small changes in the input data. The first two criteria are reasonable requirements of a sensible model of a physical situation, and the third is often expected on the basis of experimental observations. When thinking of well-posedness, we must also remember that it is often impossible to find explicit solutions to problems of practical interest, so that approximation schemes, and in particular numerical solutions, are of vital importance in practice. Hence, the question of well-posedness is intimately connected with the central question of scientific computation in partial differential equations: given the data for a problem with a certain accuracy, to what accuracy does the computed output of a numerical solution solve the problem? It is because the answer to this question is so important for modern quantitative science that well-posedness is a principal mathematical theme of this book.

Although many well-founded mathematical models of practical situations lead to well-posed problems, phenomena that are seemingly unpredictable, or at the least extremely sensitive to small perturbations, are not uncommon; examples include turbulent fluid flows described by the Navier–Stokes equations and dendrite growth modelled by the equations of solidification. Pure and applied mathematicians alike must therefore be prepared for both well-posed and ill-posed partial differential equation models. Chaos in scalar ordinary differential equations can occur if the order of the equation is at least three and so it is not surprising that what is effectively an infinite-order ordinary differential equation may have 'unpredictable' solutions. We must also remember that there are processes such as Brownian motion, which are random on a molecular scale, and yet have many properties which can be modelled by perfectly well-posed partial differential equations over much larger time and length scales. However, since we do not have the space to describe chaos theory, we will not be able to discuss the very interesting relationship between chaos and ill-posedness, although in Chapter 7 we will touch on several examples which have highly unpredictable behaviour. Nonetheless, we will be able to look at problems such as those involving exothermic chemical reactions where the model may be well-posed but its solution may only exist over a limited region in time and space before a singularity, or 'blow-up', occurs.

The advent of the computer has not only changed the attitude of the mathematical community to partial differential equations, but also the attitude of researchers in most fields where quantitative solutions of problems are now necessary. Powerful computers have encouraged people to attack so many hitherto intractable or novel problems that the number and variety of differential equations under study is increasing at an enormous rate. This observation brings us to the single most important practical reason for our writing this book, namely the 'data compression' implicit in a partial differential equation model. It is an astonishing fact that all the practical problems that we describe in this book, which range from paint flow to solidification, and from option pricing to combustion, can be described in a handful of symbols as the *quasilinear system*

$$\sum_{i=1}^{m} \mathbf{A}_i \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{b}, \tag{0.1}$$

together with suitable boundary conditions; here the unknown, $\mathbf{u}$, is a vector function of the independent variables x_i, $i = 1, \ldots, m$, while $\mathbf{A}_i$ and $\mathbf{b}$ are, respectively, square matrices and a vector which all depend on $\mathbf{u}$ and the x_i. It is the crucial fact that $\mathbf{A}_i$ and $\mathbf{b}$ do not depend on the derivatives of $\mathbf{u}$ that characterises quasilinearity. As we shall see later, we can even arrange for the right-hand side $\mathbf{b}$ to be $\mathbf{0}$.

To get some idea why this format is all-embracing, suppose we were confronted with a fairly general scalar first-order equation in two independent variables x, y in the form

$$\frac{\partial u}{\partial x} = G\left(x, y, u, \frac{\partial u}{\partial y}\right).$$

Setting $q = \partial u/\partial y$ and

$$\mathbf{u} = \begin{pmatrix} u \\ q \end{pmatrix},$$

after differentiating with respect to y, we find the system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial \mathbf{u}}{\partial x} + \begin{pmatrix} 0 & 0 \\ 0 & -\partial G/\partial q \end{pmatrix} \frac{\partial \mathbf{u}}{\partial y} = \begin{pmatrix} G \\ \partial G/\partial y + q\,\partial G/\partial u \end{pmatrix},$$

which is in quasilinear form.[1]

There is a dramatic difference between (0.1) and the ordinary differential equation when $m = 1$, namely

$$\mathbf{A}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{b}.$$

In this latter case, as long as $\mathbf{A}$ is invertible, which it usually is, and $\mathbf{A}^{-1}\mathbf{b}$ satisfies an appropriate Lipschitz condition, there is a unique solution such that $\mathbf{u} = \mathbf{u}_0$ at some point $x = x_0$. However, it is clear that if $\mathbf{u} = \mathbf{u}(x, y)$ and

$$\mathbf{A}\frac{\partial \mathbf{u}}{\partial x} = \mathbf{b},$$

then, no matter how well behaved $\mathbf{A}$ and $\mathbf{b}$ are, we cannot solve this equation with $\mathbf{u}(x, y) = \mathbf{u}_0(x)$ at $y = y_0$ unless $\mathbf{A}\,\partial\mathbf{u}_0/\partial x = \mathbf{b}$.

This observation is the basis of our discussion in Chapter 1, which concerns the scalar case of (0.1) in which the term involving the highest derivative (which is called the *principal part* of the equation) is

$$\sum_{i=1}^{m} A_i \frac{\partial u}{\partial x_i}.$$

We will begin by identifying boundary data for which we might expect a solution to exist and data for which there is almost no hope of existence. This is the theme that pervades the subsequent two chapters, which deal with systems like (0.1) and simple scalar second-order equations, respectively. We will first have to worry about ill-posedness in Chapter 2; there we shall see that when $\mathbf{u}$ is given on some initial surface, we may well be able to find all its derivatives normal to that surface but that this information only enables us to continue $\mathbf{u}$ a very small distance away from the initial surface. However, it will become apparent in Chapters 3 and 5 that this restriction can sometimes be overcome by relaxing the requirement that all components of $\mathbf{u}$ be given on this surface.

In addition to cataloguing well-behaved and badly-behaved solutions for simple scalar second-order equations, Chapter 3 also provides an introduction to Chapters 4–6, each of which deals with a class of scalar second-order equations which occurs with unfailing regularity in branches of physics, engineering, chemistry, biology, and even social science and finance. Indeed, from the practical point of view of students wanting to know how to get an analytical feel for the solutions of equations falling into these classes, these chapters form the meat of the book and can be read more or less independently.

Chapter 7 is perhaps the most unusual one in the book because it addresses a class of problems that are rarely compiled outside the research literature. Yet recent

[1] Eagle-eyed readers will notice that the first matrix is eminently invertible (because the partial differential equation has been 'solved' for $\partial u/\partial x$), while the second is not (because information is lost when we differentiate). There is a lot more to this simple calculation, as we will see in §2.3 and §8.2. By the way, because we are aiming for a concise treatment, there are many footnotes in this book, so please do not be deterred by them; they mostly contain digressions from the main stream.

inroads of mathematical modelling into practical problems, especially those arising in industry, have revealed that many, many differential equation models have to be solved in regions that are unknown *a priori*. These regions must be found as part of the solution; typical examples are the melting of an ice cube or the sloshing of water in a container. We call these problems *free boundary problems* and, in Chapter 7, we endeavour to provide an entrée into the great body of knowledge that has grown up around them in recent years.

Despite the universality of (0.1), there are some advantages in studying fully nonlinear equations in their primitive form; in Chapter 8 we revert to problems in which $\mathbf{A}$ can indeed depend on $\partial u_i/\partial x_j$ as well as on $\mathbf{u}$. Thus (0.1) is no longer quasilinear and we will see that this means that we always encounter the possibility of non-existence or non-uniqueness when we attempt to find the derivatives of $\mathbf{u}$ in terms of its values on some known surface. This will be found to lead to many fascinating generalisations of the theory of non-quasilinear ordinary differential equations, such as envelope solutions and caustics, which means that geometric interpretations are even more valuable than in earlier chapters. Chapter 9 is a compendium of ideas concerning partial differential equations that do not fit conveniently into the earlier chapters: it could have gone on for ever.

One crucial mathematical idea that will emerge from the first six chapters is the value of being able to write down formally the solution of any *linear* partial differential equation, i.e. one in which $\mathbf{A}_i$ are independent of $\mathbf{u}$ and $\mathbf{b}$ is linearly dependent on $\mathbf{u}$ in (0.1). This idea is a generalisation of the one that says that, in order to solve a system of linear algebraic equations, we have to invert a matrix; instead of writing that $\boldsymbol{\mathcal{A}}\mathbf{x} = \mathbf{b}$ usually implies $\mathbf{x} = \boldsymbol{\mathcal{A}}^{-1}\mathbf{b}$, we say that $\mathcal{L}\mathbf{u} = \mathbf{f}$ usually implies $\mathbf{u} = \mathcal{L}^{-1}\mathbf{f}$. We will see that '$\mathcal{L}^{-1}$' can, when it exists, be expressed as an integral weighted by what is called a Green's function or Riemann function. However, finding this function or even some of its simple properties is almost always difficult and usually impossible. Hence readers should never be lulled into thinking that, because of their apparent conceptual simplicity, linear partial differential equations are either easy or boring.

There is one other remark we must make before we start. This is the regrettable fact that, in order to keep this book as short as it is, we have had to exclude almost all discussion of functional analysis, numerical methods, and in particular almost all discussion of the multitude of results that can be obtained by 'perturbation theory'. In fact, we will restrict attention to those results that can fairly easily be proved analytically or interpreted geometrically. It would have been easy in principle to double the length of most of the chapters by appending some of the important results that emerge from the relevant perturbation theory; it could have been doubled again had numerical methods been included, and yet again by describing the principal results from the modern function-analytic theory of partial differential equations. However, we emphasise that many of the results we obtain or cite would not have been discovered had not their originators experimented with approximate methods at the start.

Another advantage of our self-imposed restrictions is that the only prerequisites we hope the reader possesses are some familiarity with the idea of ordinary differential equations, functions of a single complex variable and the calculus of functions of several real variables. Most of all we would like them to know the Fredholm Alternative, but, in case this is unfamiliar, it is spelled out on p. 43. Although these are not demanding prerequisites, it will help if the reader can also bring to the book a relaxed mathematical attitude and a readiness to look at the broader picture: this is not a 'definition–theorem–proof' book, nor an exhaustive catalogue of methods and techniques. The authors' background is in physical applied mathematics, which inevitably slants some of the motivational examples and interpretations of the theory, but the basic message of well-posedness would have been the same had they been numerical analysts or probabilists. The fact that we have been able to eschew rigour and relegate certain calculations to the exercises means that we have been able to keep the book relatively short without compromising its applicability.

$*$ To make a first reading easier, we have marked the harder sections of the text and exercises with an asterisk, so that they can be freely ignored by those who are pressed for time.

A bibliography, which consists almost entirely of related textbooks, is provided after Chapter 9.

1

First-order scalar quasilinear equations

1.1 Introduction

Even though this chapter deals only with the simplest category of partial differential equations, the theory that emerges is relevant to many important and fascinating practical situations. An example is the flow of a thin coat of paint down a wall, as illustrated in Fig. 1.1(a). Because the layer is thin, the velocity, say $u(x, y, t)$, is approximately unidirectional down the wall. Gravity is resisted by the viscosity of the paint, resulting in a shearing force, which we assume to be proportional to the velocity gradient $\partial u/\partial y$. A force balance on a small fluid element then shows that $\partial^2 u/\partial y^2$ is a constant, $-c$, which is proportional to gravity (see Fig. 1.1(b)). We assume that the paint sticks to the wall, so $u = 0$ on $y = 0$. Also, since the shearing force is zero on the paint surface $y = h(x, t)$, $\partial u/\partial y = 0$ there, and hence

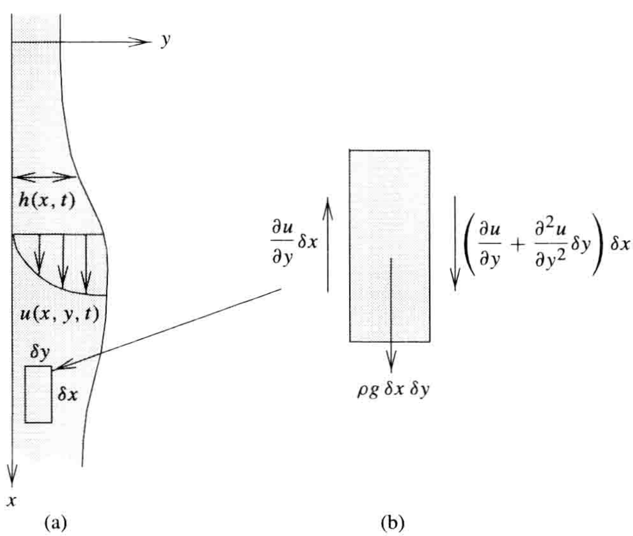

Fig. 1.1 (a) Paint on a wall. (b) Forces on a fluid element.