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HYDRODYNAMIC INSTABILITY

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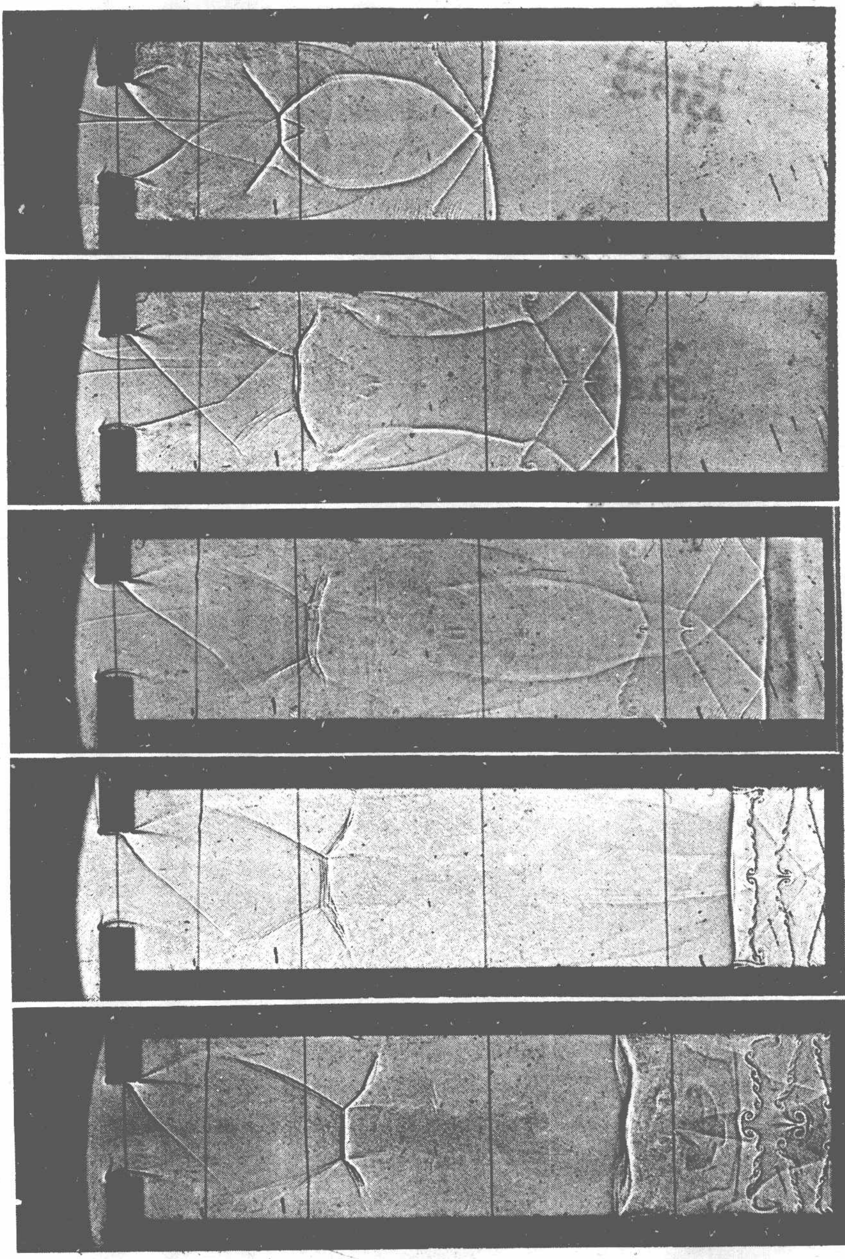

A

B

C

D

E

Diffraction of a Mach 2.1 shock wave, showing Helmholtz instability of slip lines (see p. 77)

FOREWORD

This volume deals with the following loosely related subjects: hydrodynamic instability, the statistical theory of turbulence, water waves, waves in a medium with random properties, and the theory of guidance and control. These subjects all use similar mathematical methods and ideas, and it seemed impossible to give an adequate picture of contemporary views about hydrodynamic instability without touching on them all. The fact that such a diversified group of problems are exhibited in hydrodynamics alone can only make one admire the inexhaustibility of fascinating problems that occur in nature. Yet we have hardly tapped on the range of problems that arise in astrophysics.

Even a casual perusal of the papers below will bring out a steady trend in theories of hydrodynamic instability, from the linear deterministic processes analyzed so successfully by Kelvin and Rayleigh, to *nonlinear random processes* whose theory must be regarded as the primary goal of future research efforts. We hope this book will help to stimulate and guide such efforts.

The editors wish to thank the authors for providing an excellent collection of papers dealing with such a diversified group of topics and for their cooperation in the publication of this volume.

G. BIRKHOFF
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CONTENTS

ON THE INSTABILITY OF SHEAR FLOWS

BY

C. C. LIN AND D. J. BENNEY

1. **Introduction.** One of the most basic and challenging problems in fluid mechanics is to reach an understanding of the various physical mechanisms involved in the process of transition from laminar to turbulent flow. It is now well known that in many instances the initial process is one of instability with respect to infinitesimal disturbances. This initial disturbance could take on the form of a steady secondary flow, as in the case of convective instability, or the form of a small oscillation, as in the case of certain shear flows. It is quite likely that the processes governing the subsequent development of such disturbances are also different. The present note deals with the case of shear flows.

We consider a small disturbance of the basic flow by writing the velocity and pressure of the disturbed motion in the form

$$(1.1)\qquad \begin{aligned} u_i(x_k,t) &= u_i^{(0)}(x_k) + au_i^{(1)}(x_k,t) + a^2u_i^{(2)}(x_k,t) + \cdots, \\ p(x_k,t) &= p^{(0)}(x_k) + ap^{(1)}(x_k,t) + a^2p^{(2)}(x_k,t) + \cdots \end{aligned}$$

$(i,k = 1,2,3)$ where a is a measure of the amplitude of the disturbances and all the variables are dimensionless. The above expressions must satisfy the equation of continuity and the Navier–Stokes equations, from which we obtain a sequence of equations governing the motion of each order. The equations of order a^0 give the basic flow, those of order a give the usual linear theory, and those of the order a^2 give the nonlinear effects under consideration.

In the case of shear flow, the linear terms give rise to stable and unstable normal modes in the form of waves. There is now ample evidence to support the correctness of the theory. The mathematical basis of the theory involves, however, many subtle points, especially with respect to the relationship between the solutions at finite Reynolds numbers and the inviscid limit. The uniformly valid asymptotic solutions for large Reynolds numbers have been studied by Wasow [5], by Langer [6] and by Lin and Rabenstein [7]. Other issues have been investigated by Case [8] and by Lin [9]. The major results of these investigations will be discussed in §§2, 3.

The nonlinear effect has been examined theoretically by Meksyn and Stuart [1], and later by Stuart [2], using a somewhat different approach. A theory of secondary instability has also been proposed by Görtler and Witting [3]. Experimental observations, especially those by Schubauer, Klebanoff and Tidstrom [4], however, show remarkable characteristics in the behavior of the flow not accountable by the theories of these authors. In particular, the experimental observations indicate the existence of longitudinal vortical motions (with axis along the direction of the basic flow) which give a redistribution of the momentum of mean flow in the plane normal to it. Such a vortical motion is very similar to that calculated by the Görtler–Witting theory, but its location relative to the primary oscillation is found to be exactly opposite in phase to the theoretical prediction. The theories of Meksyn and Stuart and of Stuart are based on two-dimensional primary oscillations and therefore fail to provide for such motions.

In the present theory, we examine the general equations governing the terms of the second order for a first-order primary oscillation whose amplitude varies periodically in the cross-wind direction. We found that the nonlinear effects naturally divide themselves into two categories: (1) the "two-dimensional" effects examined by Meksyn and Stuart with a variation in intensity in the spanwise direction, and (2) the intrinsically three-dimensional effects which include the longitudinal vortical motions. Detailed calculations were made with a basic profile

$$u_1^{(0)}(x_k) = U(y) = \tanh y.$$

It is found that the combined effect of the nonlinear longitudinal vortical motion and that due to the primary oscillations yields a net motion similar to that observed by Schubauer, Klebanoff and Tidstrom. The vortices are found to be at points where the primary oscillation yields a convex streamline. Other general features of the flow are also similar between the theoretical and the experimental results. The agreement is all the more remarkable since there is a substantial difference in the basic profile between the theoretical and experimental cases. It is believed that this indicates the generality of the features revealed by these investigations.

Another general feature is the important role of the critical layer. It is found that the secondary motion is generally larger than the order of a^2 by factors involving powers of $(\alpha R)^{1/3}$ where R is the Reynolds number of the basic flow. This is the familiar parameter that occurs with inner friction layers. Thus, the effect of these second-order terms becomes large through the weakness of the critical layer; but once this large secondary flow occurs, its magnitude remains of the same order even outside of the critical layer. The importance of the critical layer in the transition process has been conjectured in a previous discussion of other nonlinear effects. A brief description of these nonlinear effects found from the theoretical investigations and their comparison with experiments will be presented in §§4–6.

2. **The linear theory.** We shall now discuss the linear theory of hydrodynamic stability especially in connection with the behavior of the solution at large Reynolds numbers. We consider the stability of a parallel flow, for example, the pressure flow through a channel between parallel plates placed at $y = \pm 1$. The flow is in the x-direction with a velocity distribution $U(y) = 1 - y^2$ (or some other parabolic function). The equation for a two-dimensional small disturbance is

$$\frac{\partial \zeta'}{\partial t} + U\frac{\partial \zeta'}{\partial x} + v'\frac{d^2U}{dy^2} = \nu \Delta \zeta', \tag{2.1}$$

where ζ' is the disturbance vorticity, related to the disturbance stream function $\psi'(x,y,t)$ by

(2.2) $$\zeta' = \frac{\partial^2\psi'}{\partial x^2} + \frac{\partial^2\psi'}{\partial y^2} = \Delta\psi',$$

and the velocity components are given by

(2.3) $$u' = \frac{\partial\psi'}{\partial y}, \qquad v' = -\frac{\partial\psi'}{\partial x}.$$

The constant ν is the kinematic viscosity coefficient, or the inverse of the Reynolds number in the present dimensionless formulation. The boundary conditions are

(2.4) $$u' = v' = 0 \quad \text{at} \quad y = \pm 1.$$

The solution of (2.1) can be treated either in terms of the theory of normal modes or as an initial value problem. In the first approach, we superpose particular solutions of the form

(2.5) $$\psi'(x,y,t) = \text{Re}\,\{\phi(y)e^{i\alpha(x-ct)}\},$$

where $\phi(y)$ satisfies the familiar Orr–Sommerfeld equation

(2.6) $$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i(\alpha/\nu)[(U-c)(\phi'' - \alpha^2\phi) - U''\phi],$$

which is to be solved with the boundary conditions

(2.7) $$\phi(\pm 1) = \phi'(\pm 1) = 0.$$

In the initial value approach used by Case [8], we consider the Laplace transform of $\psi'(x,y,t)$ with respect to t and its Fourier transform with respect to x (α real, $\text{Re}\,(p) > 0$):

(2.8) $$\bar{\psi}'(y;\alpha,p) = \int_0^\infty \int_{-\infty}^\infty \psi'(x,y,t)\cdot e^{i\alpha x}dx \cdot e^{-pt}dt.$$

After solving for $\bar{\psi}'(y;\alpha,p)$ in terms of the initial conditions and the boundary conditions, we calculate $\psi'(x,y,t)$ by the inverse transform

(2.9) $$\psi'(x,y,t) = \int\int \bar{\psi}'(y;\alpha,p)\cdot e^{-i\alpha x}d\alpha \cdot e^{pt}dp,$$

where the integration with respect to p is taken along the line $\text{Re}\,(p) = p_0 > 0$ in the direction of increasing $\text{Im}\,(p)$. The normal mode representation is obtained by evaluating the p-integral at the singularities of $\bar{\psi}'$ in the p-plane by the theory of residues.

A natural approach to the solution of (2.1) for the case of small viscosity is to begin with the case $\nu = 0$. However, since the resultant equation contains spatial derivatives of lower orders, the perturbation procedure is singular, and one has to guard against all the associated pitfalls that may occur. For example, the complete equation (2.6), being regular, has only *discrete* eigenvalues; whereas the reduced equation

(2.10) $$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = 0,$$

in general has a singularity at $y = y_c$ where $U = c$, and therefore possesses *continuous* eigenvalues when the boundary conditions

$$\phi(\pm 1) = 0 \tag{2.11}$$

are imposed. The question may then be raised: in which sense does the inviscid limit represent an adequate approximation to the physical situation (which is after all always viscous)?

An obvious answer is the following. The solutions of the boundary value problem (2.1) fall into two classes: (1) those which approach the inviscid solution in the limit $\nu \to 0$; and (2) those which do not. *A priori*, either class may be empty; but closer examination reveals that there are important solutions in both classes. The members of the first class, which are expected to exhibit the boundary layer behavior, can be further divided into two sub-classes: (a) those with a boundary layer thickness of the order of $\nu^{1/2}$, and (b) those with some other thickness. The initial value method easily yields solution of class (a), whereas solutions of class (b), with a boundary layer thickness of the order of $\nu^{1/3}$, have been known through the method of normal modes. It is noteworthy that the unstable disturbances responsible for the initiation of turbulence in the channel or the boundary layer is of this latter type.

It is perhaps illuminating to examine a simple example to illustrate the variety of behavior of solutions when a process of singular perturbation is involved. Consider the equation

$$\frac{\partial u}{\partial t} + k\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (k > 0), \tag{2.12}$$

to be solved in the region $0 \leqq y \leqq 1$ with the boundary conditions $u(0) = u(1) = 0$. The inviscid problem is

$$\frac{\partial u_0}{\partial t} + k\frac{\partial u_0}{\partial y} = 0, \tag{2.13}$$

with no boundary conditions required. The normal modes (solutions obtained by the method of separation of variables) of the viscous problem are the *discrete* set of functions

$$U_n(y,t,\nu) = w_n(y)e^{p_n t}, \tag{2.14}$$

where

$$\begin{aligned} w_n &= e^{ky/2\nu}\sin n\pi y, \\ p_n &= -\frac{k^2}{4\nu} - \nu n^2\pi^2, \end{aligned} \tag{2.15}$$

and $n = 0, 1, 2, \cdots$. None of these eigenfunctions has a limit as $\nu \to 0$. On the other hand, the inviscid problem has the normal modes

$$u_0 = e^{\alpha y}e^{-k\alpha t}, \tag{2.16}$$

with continuous spectrum for the eigenvalue α. It is clear that if we attempt to solve the complete equation (2.12), with the initial condition $e^{\alpha y}$, we should get a solution $u(y,t,\nu)$ which approaches the above inviscid solution (2.16) as $\nu \to 0$. On the other hand, the representation of this solution in terms of the normal modes $u_n(y,t,\nu)$ is of the form

$$u(y,t,\nu) = \sum_{1}^{\infty} A_n(\nu)w_n(y,\nu)e^{p_n(\nu)t}, \tag{2.17}$$

where we have emphasized the dependences of A_n, w_n and p_n on ν. It is obviously a complicated problem if one tries to calculate the limit of $u(y,t,\nu)$ as $\nu \to 0$ from the above series.

The above example illustrates the following. *A normal mode in the inviscid theory may not be the limit of a normal mode in the viscous theory. Conversely, a normal mode in the viscous theory may not have an inviscid limit.* This exemplifies the classification of solutions into the two classes (1) and (2) above. The example is, however, too simple to illustrate the further subdivision of solutions of class (1).

We shall not go into further discussion of the relative merits of the method of normal modes versus the method of initial values—which is made elsewhere. Since we are going to hear from Professor Case on the method of initial values, we shall proceed to the method of normal modes, which depends on the solution of the eigenvalue problem (2.6). We are especially interested in the cases where ν is small.

3. **Asymptotic solutions of the Orr–Sommerfeld equation.** It is well known that the equation (2.6) has formal asymptotic solutions of the forms

$$\phi = \phi^{(0)} + \frac{1}{\alpha R}\phi^{(1)} + \cdots \tag{3.1}$$

and

$$\phi = \exp{(\alpha R)^{1/2}Q(y)}\left[f^{(0)} + \frac{1}{(\alpha R)^{1/2}}f^{(1)} + \cdots\right], \tag{3.2}$$

$$Q(y) = \int_{y_c}^{y} (i(U-c))^{1/2}dy.$$

However, these solutions have singularities at the point y_c where $U - c = 0$, whereas the original equation (2.6) is perfectly regular at that point. Search for a system of regular solutions with tractable asymptotic behavior has been made by Wasow [5], by Langer [6], and by Lin and Rabenstein [7]. The relationship between the methods used has been discussed in the last mentioned paper, the main ideas of which will now be summarized.

To put it in a more general perspective, we consider the linear differential equation of the fourth order

$$\mathscr{L}(\phi) = \frac{d^4\phi}{dx^4} + \lambda^2\left\{P(x,\lambda)\frac{d^2\phi}{dx^2} + Q(x,\lambda)\frac{d\phi}{dx} + R(x,\lambda)\phi\right\} = 0, \tag{3.3}$$

when the parameter λ is large. In the above expression all the three functions $P(x,\lambda)$, $Q(x,\lambda)$ and $R(x,\lambda)$ are analytic functions in the complex variable x, and they depend on λ in such a manner that asymptotic expansions of the following form hold:

$$F(x,\lambda) = \sum_{n=0}^{\infty} \lambda^{-n} F_n(x). \tag{3.4}$$

Especially we are interested in the behavior of the solution in the neighborhood of a *simple* zero of the function $P_0(x)$. Such a point is called a *turning point* (of the first order). It is of special interest because asymptotic solutions of (3.3) in the form

$$\phi = e^{\lambda S(x)} \sum_{n=0}^{\infty} \lambda^{-n} \phi_n(x) \tag{3.5}$$

exhibits singularities at such a point even though the point is a regular point of the differential equation (3.3). Other solution forms must therefore be found, if we wish to find a complete set of asymptotic solutions *uniformly* valid in a neighborhood containing the turning point.

We base our solution on the simplest of equations of the type (3.3); namely,

$$L(u) \equiv u^{iv} + \lambda^2(zu'' + \alpha u' + \beta u) = 0, \tag{3.6}$$

where α and β may depend on λ asymptotically in the following manner:

$$\alpha = \sum_{n=0}^{\infty} \lambda^{-n} \alpha^{(n)}, \tag{3.7a}$$

$$\beta = \sum_{n=0}^{\infty} \lambda^{-n} \beta^{(n)}. \tag{3.7b}$$

The solutions of this *basic reference equation* (3.6) can be studied by the method of Laplace transformation, as was carried out by Rabenstein [10]. Our aim is to show that we can obtain asymptotic solutions of the form

$$\phi(x) = c_0(x)u(z) + c_1(x)u'(z) + c_2(x)u''(z) + c_3(x)u'''(z), \tag{3.8}$$

where the variables z and x are connected by a suitable analytic function (e.g., (3.10) below), and the functions $c_i(x,\lambda)$ all have the asymptotic behavior (3.4). Actually, it turns out that

$$c_0, c_1 = O(1), \quad \text{and} \quad c_2, c_3 = O(\lambda^{-2}).$$

The method for establishing such a solution will be described below. For practical purposes, once we know the existence of the solutions of the form (3.8), the coefficients c_i can be calculated by using the formal asymptotic solutions of ϕ and u of the type (3.5).

The derivation of the solution (3.8) is made as follows. We first carry out *finite* transformations such that the equation (3.3) is reduced to a form as close to

(3.6) as possible (no approximation being made in this step). It can be shown that one can always achieve the *normal form*,

$$L_0(\chi) \equiv \chi^{iv} + \lambda^2(z\chi'' + \alpha_0\chi' + \beta_0\chi) = \lambda(\tilde{a}\chi + \tilde{b}\chi' + \lambda^{-1}\tilde{c}\chi'' + \lambda^{-2}\tilde{d}\chi'''), \tag{3.9}$$

where α_0 and β_0 are constants while $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ have the behavior (3.4). Such a transformation is carried out in two steps:

First transformation. We introduce into (1.1) the new independent variable,

$$z = \left[\frac{3}{2}\int_{x_0}^{x} [P_0(x)]^{1/2}dx\right]^{2/3}, \tag{3.10}$$

and the new dependent variable,

$$\psi(z,\lambda) = \phi(x,\lambda)[P_0(x)/z]^{3/4}. \tag{3.11}$$

This will have the effect of retaining the general form of the equation (1.1) but casting $P_0(x)$ into the independent variable z itself.

Second transformation. Next we consider a transformation of the form,

$$\chi = A(z,\lambda)\psi + B(z,\lambda)\psi' + \lambda^{-1}C(z,\lambda)\psi'' + \lambda^{-2}D(z,\lambda)\psi''', \tag{3.12}$$

where A, B, C, D has the behavior (3.4). By formal differentiation, we obtain the transformation in the vector form,

$$\boldsymbol{\chi} = G\boldsymbol{\psi}, \tag{3.13}$$

where

$$\boldsymbol{\chi} = (\chi,\chi',\lambda^{-1}\chi'',\lambda^{-2}\chi'''), \tag{3.14}$$

$$\boldsymbol{\psi} = (\psi,\psi',\lambda^{-1}\psi'',\lambda^{-2}\psi'''), \tag{3.15}$$

and $G(z,\lambda)$ is a nonsingular matrix with asymptotic dependence on λ in the manner,

$$G(z,\lambda) = \sum_{n=0}^{\infty} \lambda^{-n}G^{(n)}(z). \tag{3.16}$$

It can be shown by explicit calculation that this is indeed possible with a *finite* number of terms, all regular in z. In the process of carrying out this transformation, it will become clear that the constant α_0 in (3.9) is indeed completely determined by the given equation, and that the constant β_0 is also determined in the cases where α_0 is an integer, positive, negative or zero.

Construction of the asymptotic solutions. The construction of the asymptotic solutions of the normalized equation (3.9) then follows usual methods. If we introduce the vector

$$\boldsymbol{u} = (u,u',\lambda^{-1}u'',\lambda^{-2}u''), \tag{3.17}$$

where u is a solution of (3.6), then we may expect it possible, by proper determination of α and β to obtain formal solutions of the structure,

$$\chi = Hu, \tag{3.18}$$

and with H expected to be of the form,[1]

$$H = I + \sum_{m=1}^{\infty} \lambda^{-m} h^{(m)}(z), \tag{3.19}$$

where I is the identity matrix. Combining (3.13) and (3.18), we may write

$$X = G\Psi = HU, \tag{3.20}$$

or

$$\Psi = G^{-1}HU, \tag{3.21}$$

where U, Ψ, X are fundamental matrix solutions of the system of equations corresponding to the fourth order equations for u, ψ and χ. For convenience of reference, we list the equations for u and χ explicitly. The equation for u may be written

$$\frac{du}{dz} = Mu, \tag{3.22}$$

where

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ -\beta & -\alpha & -\lambda z & 0 \end{bmatrix}. \tag{3.23}$$

The equation for χ may be written as

$$\frac{d\chi}{dz} = (M_0 + \epsilon)\chi, \tag{3.24}$$

where M_0 is obtained from M by putting $\alpha = \alpha_0$, $\beta = \beta_0$, and

$$\epsilon = \lambda^{-1} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{a} & \bar{b} & \tilde{c} & \tilde{d} \end{bmatrix}, \tag{3.25}$$

in which

$$\bar{a} = \tilde{a} + (\beta - \beta_0), \qquad \bar{b} = \tilde{b} + (\alpha - \alpha_0). \tag{3.26}$$

If we terminate the series (3.19) with $h^{(n)}(z)$ as the last term, we obtain the nth approximation,

$$\chi_n = H_n u, \tag{3.27}$$

which satisfies the differential equation,

$$\frac{d\chi_n}{dz} = (M_0 + \epsilon_n)\chi_n, \tag{3.28}$$

[1] In certain special cases, fractional powers of λ may occur; but the general nature of the theory remains unchanged.

with

$$\epsilon - \epsilon_n = \lambda^{-(n+1)}\Delta, \tag{3.29}$$

where Δ is a matrix of the same form as $\lambda\epsilon$. The equation (3.28) is the approximating *related equation* of (3.24). Heuristically, it would appear natural to think of (3.7) and (3.9) as *asymptotically equivalent*, and to expect that once the construction (3.27) is made, it should be easy to prove its asymptotic validity; namely, to show that there are actual solutions of (3.24) which is approximated by (3.27). Although we found it possible to do this, the proof was *not easy*. It depends on the examination of each individual solution in a particular fundamental set of four solutions. Possibly a more general theorem can be established which corresponds more closely to heuristic expectations.

4. **The classification of nonlinear effects.** As we have mentioned earlier the linear theory gives an adequate representation of the motion when the disturbances are very small and it is also capable of predicting the onset of instability. However, as these disturbances grow the nonlinear terms in the equations of motion must be included in the analysis. For finite amplitude oscillations it is well known that two new features appear: (1) the excitation of higher harmonics of the primary oscillation and (2) a modification of the original flow profile by the action of the Reynolds stresses in producing a redistribution of momentum. Meksyn and Stuart have calculated this modification for flow between two parallel walls when the primary oscillation is strictly two dimensional.

One point of interest is the relative importance of two and three dimensional disturbances. On the basis of linearized theory, Squire's result applies. Thus, to estimate the onset of instability one need only consider two dimensional oscillations. These two dimensional waves are indeed observed experimentally during the initial instability. However, one must not be tempted to read more into Squire's theorem than it actually implies. For example, at a Reynolds number beyond the critical Reynolds number there will be unstable three dimensional waves and there is no guarantee that the most highly amplified disturbance will still be two dimensional, even on the basis of linear theory. The details would depend on the particular profile. Indeed the simple observation that turbulence is an essentially three dimensional phenomenon suggests that this two dimensional supremacy cannot be expected to persist through the finite amplitude regime. Recent experimental evidence (to which we will refer later) points strongly to the desirability of a theoretical investigation of three dimensional disturbances.

We now examine finite amplitude oscillations in a given parallel flow, paying special attention to the three dimensionality. This is done by a straightforward perturbation from the linear theory as we shall describe below.

If p, $\boldsymbol{q}$, $\boldsymbol{\omega}$ denote the pressure, velocity and vorticity respectively and we set up the perturbation (suggested in the introduction) by writing

$$p = p^{(0)} + ap^{(1)} + a^2p^{(2)} + \cdots, \tag{4.1}$$

(4.2) $$q = q^{(0)} + aq^{(1)} + a^2q^{(2)} + \cdots,$$

(4.3) $$\omega = \omega^{(0)} + a\omega^{(1)} + a^2\omega^{(2)} + \cdots,$$

where $q^{(0)}$ is the original basic flow, $q^{(1)}$ the primary oscillation, etc. The symbol a is used to denote a perturbation amplitude, and $p^{(0)}$ is the pressure distribution associated with the basic flow. We take xyz as rectangular coordinates in a given parallel basic flow $q^{(0)} = (u_0^{(0)}(y),0,0)$, and consider waves of small amplitude propagating downstream having a possible z variation of amplitude. We write

(4.4) $$q^{(n)} = (u^{(n)},v^{(n)},w^{(n)}),$$

(4.5) $$\omega^{(n)} = (\xi^{(n)},\eta^{(n)},\zeta^{(n)}) = \text{curl } q^{(n)}.$$

Equating successive powers of a to zero in the continuity equation and in the Navier–Stokes equations we have

(4.6) $$\text{div } q^{(n)} = 0,$$

(4.7) $$\frac{\partial q^{(n)}}{\partial t} + \sum_{r=0}^{n} (q^{(r)} \cdot \nabla)q^{(n-r)} = -\nabla p^{(n)} + \frac{1}{R}\Delta q^{(n)}.$$

If α is the downstream wave number of the primary oscillation, we may write,

(4.8) $$p^{(1)} = p_1^{(1)}(y,z,t)e^{i\alpha x} + p_1^{(1)*}(y,z,t)e^{-i\alpha x},$$

(4.9) $$q^{(1)} = q_1^{(1)}(y,z,t)e^{i\alpha x} + q_1^{(1)*}(y,z,t)e^{-i\alpha x},$$

(4.10) $$\omega^{(1)} = \omega_1^{(1)}(y,z,t)e^{i\alpha x} + \omega_1^{(1)*}(y,z,t)e^{-i\alpha x},$$

where an asterisk denotes a complex conjugate. The equations governing the first order motion are

(4.11)
$$i\alpha u_1^{(1)} + \frac{\partial v_1^{(1)}}{\partial y} + \frac{\partial w_1^{(1)}}{\partial z} = 0,$$
$$\frac{\partial u_1^{(1)}}{\partial t} + i\alpha u_0^{(0)}u_1^{(1)} + v_1^{(1)}\frac{du_0^{(0)}}{dy} = -i\alpha p_1^{(1)} + \frac{1}{R}\left(\frac{\partial^2}{dy^2} + \frac{\partial^2}{dz^2} - \alpha^2\right)u_1^{(1)},$$
$$\frac{\partial v_1^{(1)}}{\partial t} + i\alpha u_0^{(0)}v_1^{(1)} = \frac{-\partial p_1^{(1)}}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2\right)v_1^{(1)},$$
$$\frac{\partial w_1^{(1)}}{\partial t} + i\alpha u_0^{(0)}w_1^{(1)} = \frac{-\partial p_1^{(1)}}{\partial z} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \alpha^2\right)w_1^{(1)}.$$

The second order motion which we shall refer to as the secondary flow can be written as the sum of a mean motion and a second harmonic oscillation. Superscripts have been used to denote the order of the motion and subscripts for the harmonic content. Thus we have

(4.12) $$p^{(2)} = p_0^{(2)}(y,z,t) + p_2^{(2)}(y,z,t)e^{2i\alpha x} + p_2^{(2)*}(y,z,t)e^{-2i\alpha x},$$

(4.13) $$q^{(2)} = q_0^{(2)}(y,z,t) + q_2^{(2)}(y,z,t)e^{2i\alpha x} + q_2^{(2)*}(y,z,t)e^{-2i\alpha x},$$

(4.14) $$\omega^{(2)} = \omega_0^{(2)}(y,z,t) + \omega_2^{(2)}(y,z,t)e^{2i\alpha x} + \omega_2^{(2)*}(y,z,t)e^{-2i\alpha x}.$$