LINEAR ORDERINGS

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For Judy
and for Mira and Ariela
and now for Dalia too

In memory of

my father's mother
Dvorah (Kupritz) Rosenstein
and my mother's brothers and sisters and their families
Moshe Aaron Kaganowicz, his wife, Gitel, and daughter, Sonya
Blume (Rubinstein) and her children, Reuven and Shifra
Mordechai (Motel), his wife, Esther, and daughter, Aliza
Miriam (Mira), her husband, Simcha, son, Zalman, and another child
Batye (Baranchik), her husband, and three or four children
Malkah (Kravetz)
and many other members of my family who died in the Holocaust

Preface

A book about linear orderings? You mean total orderings? What can you possibly say about them? After all, besides the natural numbers, the integers, the rationals, and the reals, what linear orderings are there?

These questions, usually unspoken, were common. It is my hope that the reader will find this book a satisfactory response.

My interest in linear orderings was first aroused by an old paper of Dushnik and Miller. Perhaps you too will find the following question interesting:

> Given a well-ordering A, it is easily verified that any order-preserving map from A to A (that is, $f(a) < f(b)$ whenever $a < b$) is also non-decreasing (that is, $f(a) \geq a$ for all a in A.) Is the converse true? That is, given that any order-preserving map from A to A is non-decreasing, does it follow that A is a well-ordering?

They answered this question using a simple, elegant technique. I observed that a number of theorems about linear orderings (including one of my own) were proved using variations of this technique, which I have called "condensing linear orderings." That many different facts about linear orderings share an underlying theme suggested that there is a subject called *linear orderings* which is more than just a collection of isolated facts.

As I examined the literature, I became more convinced of the usefulness of presenting the extensive material on linear orderings in a unified manner. I found that the various groups of people who studied linear orderings were generally unaware of other people's work. I also found that the lack of a comprehensive treatment resulted in theorems being re-proven (like Hausdorff's Theorem 5.4) and techniques being rediscovered (like that of Dushnik and Miller's Theorem 9.1).

My study of the literature also convinced me that there was too much material for one book. It took a while for me to become convinced of this, and I have persisted in incorporating more and more material into this book. However, I have learned (with apologies to Koheleth) that of writing a book there is no end—and so this book is now, thank God, completed, and it contains just what it contains.

My original goal was to include everything known about linear orderings not already in Sierpinski's *Cardinal and Ordinal Numbers*. At a later time, my more modest goal was that the book should contain what every logician would want to know about linear orderings. Still later, although I no longer had a clearly formulated goal, I knew exactly what material the book would contain. In retrospect one might say that, like any author, I put into the book precisely the material that most interested me.

The reader who wishes to look further will find bibliographies attached to each of the later chapters and a rather complete bibliography of all articles on linear orderings (including those not referred to in the text) at the very end.

What I have included can be seen from the table of contents. The book divides naturally into three parts: Introduction to Linear Orderings (Chapters 1–3), Combinatorial Aspects of Linear Orderings (Chapters 4–11), and Logical Aspects of Linear Orderings (Chapters 12–16).

The introductory part contains some material with which every mathematician should be familiar and other material which, though introductory, is less familiar. By including this material the book becomes essentially self-contained and can be used as a textbook for a course on linear orderings—at either an undergraduate level or graduate level, for either combinatorialists or logicians. Combinatorialists will find that the first two parts of the book are completely self-contained. For those combinatorialists who would like to read on and see why logicians are interested in linear orderings, I have provided an introductory chapter on mathematical logic.

Having said that the book is self-contained, I must immediately backtrack a little bit. The natural context, from a mathematical point of view, for discussing ordinals is axiomatic set theory, but presenting that context would make this book too long and, more seriously, would deflect the reader from getting into the subject of linear orderings. I therefore avoid introducing axiomatic set theory and instead assume a naïve familiarity with notions of set theory (like cardinal numbers) and a naïve acceptance of the axiom of choice. (When a less naïve point of view is appropriate, I invoke the context of Zermelo–Fraenkel set theory with choice (ZFC).) One consequence of this position is that many results which are more set-theoretic in character unfortunately are not included in this book. This position also leads to some difficulties when, in Chapter 3, each ordinal is viewed as the set of all smaller ordinals and each cardinal number is viewed as an ordinal; but these difficulties can be resolved by any reader whose interests are foundational or overlooked by any reader whose interests lie elsewhere.

My interests in model theory also provided some incentive for writing this book. While teaching model theory, I felt that there was a dearth of concrete examples illustrating the basic notions, and so I was led to investigate various classes of structures. I found that linear orderings and partial orderings served well as testing grounds for model-theoretic concepts and conjectures. The examples and observations I found illustrative I have included in the appropriate chapters. As a result, Chapter 13 can be used, together with certain earlier material, for an introductory course in model theory. Similarly, Chapter 16 can be used for an introductory course in recursive function theory.

Essentially none of the non-introductory material has ever appeared in a book; much of the research reported is relatively recent. I have tried to attribute concepts and results to their creators or discoverers; any lapses I sincerely regret. The non-introductory sections of Chapter 16 consist largely of my own results; other unpublished material has been incorporated into various chapters.

The book contains a large number of exercises. Although it is common to distinguish between easier and more difficult exercises, I have chosen not to do so—leaving that too as an exercise. All unproved lemmas and theorems in the first three chapters (and occasionally elsewhere) are indicated by a ▲ rather than the ■ used at the end of a real proof; these should also be treated as exercises and verified by the reader.

Books are meant to be read, and mathematics books should not be exceptions. I have tried to write clearly and discursively, attempting to reveal what the notation often conceals. In this I have tried to follow Sierpinski's example. It is for the reader to judge whether this attempt was successful.

Acknowledgments

A large part of this book was written during the spring semesters of 1976 and 1977, when I was a Member of the Institute of Advanced Study, and during the summer of 1978, when I was a visitor there. I would like to take this opportunity to thank the Institute and its staff for providing the facilities and the services which helped make this book possible.

A number of people have assisted me by reading parts of this book and offering their suggestions, comments, criticisms, and encouragement. Peter Mulhall read the first two parts of the book, and his suggestions are reflected throughout the book. Charles Landraitis read a number of chapters and offered valuable comments. I would also like to thank G. Cherlin, S. Fellner, R. Fraïssé, F. Galvin, A. Glass, Y. Gurevich, J.-G. Hagendorf, C. Holland, R. Laver, and J. Schmerl for reading and commenting on various chapters of the book.

I suppose that those errors which remain are my responsibility, but I would rather fault some gremlins. I will be grateful if the reader would bring any evidence of their interference to my attention.

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PART I

INTRODUCTION TO LINEAR ORDERINGS