INTEGRAL REPRESENTATIONS OF FUNCTIONS AND IMBEDDING THEOREMS

Volume I

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INTEGRAL REPRESENTATIONS OF FUNCTIONS AND IMBEDDING THEOREMS

Volume I

TRANSLATION EDITOR'S PREFACE TO VOLUME I

A major trend in contemporary mathematics, for a period exceeding 50 years, has been the study of spaces of functions that satisfy difference conditions (such as the Hölder continuity) and functions that satisfy differentiability conditions, *plus* the imbedding relations among and between these various spaces. From the beginning of this study, the Russian school has been a central contributor and, in recent years, their acknowledged leader has been Sergey M. Nikol'skiĭ. In his 1969 book Nikol'skiĭ summarized the contributions of his school, using approximation by entire functions of exponential type as his main tool. In this book, Nikol'skiĭ and his colleagues Valentin P. Il'in and Oleg V. Besov bring us up to date. Integral representations using kernels that are adapted to the "shape" of the domain of the function constitute the main tool used in *Integral Representations of Functions and Imbedding Theorems.*

The Russian text was written in a somewhat informal style and we have attempted to preserve the liveliness of the original. As the translation editor, I should like to add a personal note. A substantial part of my early mathematical work was built on the studies described in this book. In particular, the pioneering work of what most properly are known as Besov spaces was fundamental to my studies. I hope my efforts on this edition will repay, in part, my debt to Oleg V. Besov and his co-workers.

There are two volumes in the English-language edition of *Integral Representations of Functions and Imbedding Theorems.* The first three chapters appear in Volume I and the last three in Volume II. Chapter 1 is concerned with various integral inequalities, and in particular with a version of the Calderón–Zygmund theory in §4. Chapter 2 introduces the major idea of the book, integral representations. In Chapter 3 the authors introduce anisotropic Sobolev spaces on domains that satisfy a horn-condition (a generalization of a cone-condition) and study imbedding theorems among these spaces.

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INTRODUCTION

The theory of imbedding of spaces of differentiable functions of several real variables developed as a new trend in mathematics in the 1930's as a result of the works of S. L. Sobolev. In 1950, he organized these results in the form of a monograph [2]. This theory studies important connections and relationships of differentiability properties of functions in different metrics. Apart from its independent interest from the point of view of the theory of functions, it has numerous effective applications in the theory of partial differential equations. Sobolev included such applications in his monograph. He studied isotropic spaces $W_p^{(l)}(G)$ of functions $f(x)$ defined on a region G contained in E^n with norm

$$\sum_{|\alpha| \leqslant l} \| D^\alpha f \|_{p,\, G},$$

where l is a natural number, $p \geqslant 1$, and $\| f \|_{p,G} = \left\{ \int\limits_G | f(x) |^p \, dx \right\}^{1/p}$.

Sobolev obtained the first imbedding theorems for regions in n-dimensional spaces, specifically, the theorems on q-summability of the derivatives $D^\beta f$ over a region or over manifolds of lower dimension contained in a region.

In recent years, imbedding theory has been intensively developed in various directions by the efforts of many mathematicians and has acquired new interesting and important applications.

S. M. Nikol'skiĭ developed a theory of imbedding of spaces

$$H_p^l(E^n), \quad l=(l_1, \ldots, l_n), \quad 1 \leqslant p \leqslant \infty,$$

the functions in which are characterized, first, by differential indices of both integral and nonintegral orders (Hölder conditions) and, second, by the fact that, in general, they have different properties with respect to the different variables. He first obtained a generalization of those imbedding theorems that deal with a restriction to manifolds of lower dimension. The method that he used was based on approximation of functions by trigonometric polynomials or entire functions of the exponential type (see [1], [9]).

The first definitive results in the problem of traces of functions in Sobolev spaces were obtained for $p = 2$ by Aronszajn [1] and independently by V. M. Babič and L. N. Slobodeckiĭ [1] and by Freud and Kralik [1]. Slobodeckiĭ [2] developed for $p = 2$ a complete theory of anisotropic Sobolev spaces $W_2^l(E^n)$, where $l=(l_1, \ldots, l_n)$, with both integral and fractional indices of differentiation of functions. Gagliardo [1] characterized, for $1 \leqslant p < \infty$, the traces of functions in the Sobolev space $W_p^{(1)}(E^n)$ on an $(n-1)$-dimensional cross-section of E^n.

O. V. Besov [3] used approximation methods to develop a theory of the spaces $B_{p,\theta}^l(E^n)$, which are interesting in that they, like H_p^l-spaces, form a closed system with respect to the imbedding theorems and also have a close connection with Sobolev (and Slobodeckiĭ) spaces since, for a suitable choice of the parameters, they coincide with $W_2^l(E^n)$ and also with the spaces of traces on E^m (for $m < n$) of functions in $W_p^l(E^n)$, where $1 < p < \infty$.

Sobolev proved his imbedding theorems by means of integral representations of functions in terms of their derivatives. This method of integral representations was then further developed in the works of V. P. Il'in and, in particular, they were carried over to cases of representation in terms of differences. One of the important advantages of the method of integral representations is that the representation of a function at a given point x is constructed from the values of that function at points of a bounded cone (or horn) with vertex at the point x. This made it possible to study function spaces of functions defined on an open set of rather general form (a star-shaped region with respect to a ball, an open set with a cone condition or with an l-horn condition, etc.).

In the development of various aspects of the theory of imbedding of function spaces presented in the present book, other mathematicians made their contributions: P. I. Lizorkin, S. V. Uspenskiĭ, K. K. Golovkin, V. A. Solonnikov, V. I. Burenkov, and others. References to their contributions will be made in the course of the exposition.

The monograph of S. M. Nikol'skiĭ [9] appeared in 1969. Among other questions, this monograph illuminated a certain aspect of the theory of imbedding of spaces of differentiable functions. It was devoted primarily to the study of functions and function spaces defined on the entire n-dimensional Euclidean space. In it, an instrument of study is the apparatus of approximation of functions by means of entire functions of the exponential type.

The present book and Nikol'skiĭ's monograph can be regarded as two parts of a single work, presenting the results of the development of the basic trends in imbedding theory over a period of many years.*

These two books differ both in approach and in subject matter. Here, the basic apparatus is integral representation of functions and its subject is functions defined on regions in a Euclidean space. We shall treat anisotropic Sobolev spaces and imbedding theorems for them, various families of spaces of functions characterized by difference relations, the behavior of differentiable functions of particular classes at infinity, estimates of mixed derivatives in terms of differential operators, the dependence of imbedding theorems on the structure of a region, a generalization of the Zygmund-Calderón theorem on estimates of singular integrals together with applications

*Familiarity with Nikol'skiĭ's monograph is not a prerequisite for the present book.

of it, traces of functions on manifolds, questions of compactness of sets of functions, and classes of the Morrey and Campanato types.

We do not discuss the theory of imbedding of weighted function spaces. We mention only that the ideas and methods expounded have direct and broad applications in that theory. The book is designed for readers familiar with the Lebesgue integral.

The numbering of the formulas begins afresh in each section (indicated by §). If reference is made to a formula in the section in which the formula occurs, it will be indicated by the formula number in parentheses, for example (39). If the formula is in another section, the formula number in parentheses is preceded by the number of the section in which the formula occurs, for example, 2(17). The numbers of the theorems, lemmas, etc., coincide with the numbers of the subsections in which they appear. They are referred to by the subsection number.

In conclusion, the authors consider it their pleasant duty to express their deep gratitude to Viktor Ivanovič Burenkov and Petr Ivanovič Lizorkin, who read the book in manuscript and made a number of valuable comments. Many of these were followed and they made possible an improvement of the book.

Chapter 1

INTEGRAL INEQUALITIES

An important part of the investigation tools used in the present book is constituted by estimates of integral operators of different kinds. These estimates rest primarily on classical integral inequalities such as those of Hölder and Minkowski, the generalized Minkowski inequality, Hardy's inequality, the Hardy-Littlewood inequality for fractional integrals, the Mihlin-Calderón-Zygmund inequality for singular integrals, and various generalizations of them.

In the present chapter, we shall present the basic integral inequalities to be used later on and we shall also present the necessary information on L_p spaces of real functions.

§1. L_p spaces

1.1. In this section, we shall state certain properties of spaces $L_p(G)$ of real functions $f(x)$ defined on a measurable not necessarily bounded subset G of E^n, where E^n denotes the n-dimensional Euclidean space of points $x = (x_1, \ldots, x_n)$. Throughout, measurability should be understood in the sense of Lebesgue.

Let p denote a real number such that $1 \leqslant p < \infty$. We denote by $L_p(G)$ the space of functions $f(x)$ that are measurable on G and for which the function $|f(x)|^p$ is Lebesgue-integrable on G.

The number

$$\|f\|_{L_p(G)} = \|f\|_{p,G} = \left(\int_G |f(x)|^p \, dx\right)^{1/p}$$

is called the norm of the element $f \in L_p(G)$.

Consider also the space $L_\infty(G)$, that is, the space of functions that are measurable and essentially bounded. The norm for this space is

$$\|f\|_{L_\infty(G)} = \|f\|_{\infty,G} = \operatorname*{ess\,sup}_{x \in G} |f(x)|.$$

The notation L_∞ is justified by the fact that, for bounded G,

$$\|f\|_{\infty,G} = \lim_{p \to \infty} \|f\|_{p,G}$$

(see, for example, Nikol'skiĭ [9], 1.1).

An important subspace of the space $L_\infty(G)$ is the space $C(G)$ of functions $f(x)$ that are uniformly continuous on G with norm

$$\|f\|_{C(G)} = \sup_{x \in G} |f(x)|.$$

Thus, the spaces $L_p(G)$ are defined for all real p such that $1 \leqslant p \leqslant \infty$.

Let $\boldsymbol{p} = (p_1, \ldots, p_n)$ denote a vector with components satisfying the inequalities $1 \leqslant p_i \leqslant \infty$ for $i = 1, \ldots, n$.

We denote by $L_{\boldsymbol{p}}(E^n)$ the space of functions $f(x)$ defined on E^n that are measurable and for which the norm

$$\|f\|_{\boldsymbol{p},E^n} = \|f\|_{(p_1,\ldots,p_n),E^n} = \|\ldots\|\|f\|_{p_1,x_1}\|_{p_2,x_2}\ldots\|_{p_n,x_n} =$$

$$\left\{\int_{E^1}\left[\cdots\left\{\int_{E^1}\left(\int_{E^1}|f(x)|^{p_1}\,dx_1\right)^{p_2/p_1} dx_2\right\}^{p_3/p_2}\cdots\right]^{p_n/p_{n-1}} dx_n\right\}^{1/p_n} \tag{1}$$

is finite. We note that the order in which the norms are taken with respect to the different variables is significant; in general,*

$$\left[\int_{E^1} dx_2\left(\int_{E^1} |f(x_1, x_2)|^{p_1} dx_1\right)^{p_2/p_1}\right]^{1/p_2}$$
$$\neq\left[\int_{E^1} dx_1\left(\int_{E^1} |f(x_1, x_2)|^{p_2} dx_2\right)^{p_1/p_2}\right]^{1/p_1}.$$

Let G denote an arbitrary measurable subset of E^n and let f denote a measurable function defined on G. Then, we set

$$\|f\|_{\mathbf{p},G}=\|\tilde{f}\|_{\mathbf{p},E^n}, \tag{2}$$

where $\tilde{f}(x)=f(x)$ for $x\in G$ and $\tilde{f}(x)=0$ for $x\in E^n\setminus G$. If $\|f\|_{\mathbf{p},G}$ is finite, we write $f\in L_{\mathbf{p}}(G)$.

For simplicity, when $G=E^n$, we shall often write $\|f\|_{\mathbf{p}}$ instead of $\|f\|_{\mathbf{p},E^n}$.

We note that, if $\mathbf{p}=(p, \ldots, p)$, then

$$\|f\|_{\mathbf{p},G}=\|f\|_{p,G}.$$

In what follows, we shall write $\mathbf{p}\geqslant\mathbf{q}$ and $\mathbf{p}>\mathbf{q}$, where $\mathbf{p}=(p_1, \ldots, p_n)$ and $\mathbf{q}=(q_1, \ldots, q_n)$ to mean respectively that $p_i\geqslant q_i$ and $p_i>q_i$ for $i=1, \ldots, n$. In particular, the notation $\mathbf{1}\leqslant\mathbf{p}\leqslant\boldsymbol{\infty}$ (where $(\mathbf{1}=\overbrace{(1, \ldots, 1)}^{n}$ and $\boldsymbol{\infty}=\overbrace{(\infty, \ldots, \infty)}^{n})$ means that $1\leqslant p_i\leqslant\infty$ for $i=1, \ldots, n$.

Let us list a number of facts involving properties of spaces $L_{\mathbf{p}}(G)$ that we shall need.**

First of all, the space $L_{\mathbf{p}}(G)$, where $\mathbf{1}\leqslant\mathbf{p}\leqslant\boldsymbol{\infty}$, is a Banach space of functions with the norm defined above. In the present case, this means that the following properties hold:

*If $p_1\leqslant p_2$, then $\|f\|_{(p_1,p_2),E^2}\leqslant\|f\|_{(p_2,p_1),E^2}$ (see subsection 2.11 below).

**The article by Benedek and Panzone [1] is devoted to properties of L_p spaces.

1) $\| f \|_{p, G} = 0$ is equivalent to $f(x) = 0$ for almost all $x \in G$.
2) $\| cf \|_{p, G} = | c | \| f \|_{p, G}$.
3) $\| f_1 + f_2 \|_{p, G} \leqslant \| f_1 \|_{p, G} + \| f_2 \|_{p, G}$.
4) The space $L_p(G)$ is complete; that is, the inclusion and limit relations

$$f_k \in L_p(G) \quad (k = 1, 2, \ldots), \quad \| f_k - f_l \|_{p, G} \to 0 \text{ as } k, \quad l \to \infty, \tag{3}$$

imply the existence of a function $f \in L_p(G)$ such that $\| f_k - f \|_{p, G} \to 0$ as $k \to \infty$.

Properties 1) and 2) are obvious. Inequality 3) is known as Minkowski's inequality (it will be proven in subsection 2.7). Let us prove the completeness of the space $L_p(G)$.

Obviously, we can confine ourselves to the case $G = E^n$. Let $\{f_k\}_1^\infty$ denote a sequence of functions in $L_p(G)$ for which (3) holds.

To prove completeness, we shall use the corollary of proposition 2.6. According to the formula in that corollary, for any measurable function φ and any p such that $1 \leqslant p \leqslant \infty$, we have

$$\| \varphi \|_p = \sup_{\| g \|_{p'} = 1} \int_{E^n} | \varphi(x) g(x) | \, dx, \tag{4}$$

where $p' = (p'_1, \ldots, p'_n)$ and $\frac{1}{p_i} + \frac{1}{p'_i} = 1$ for $i = 1, \ldots, n$.

Let $\{I_m\}_1^\infty$ denote a family of bounded measurable sets such that $\bigcup\limits^{\infty} I_m = E^n$ and let χ_{I_m} denote the characteristic function of the set I_m.

Then, on the basis of (4), we have

$$\| f_k - f_l \|_p = \sup_{\| g \|_{p'} = 1} \int_{E^n} | f_k - f_l | | g | \, dx \geqslant \| \chi_{I_m} \|_{p'}^{-1} \int_{I_m} | f_k - f_l | \, dx.$$

By virtue of (3) and the completeness of the space L_1, this inequality implies the existence of a function f defined on E^n such that $\|(f - f_k)\chi_{I_m}\|_1 \to 0$ for every m as $k \to \infty$. By means of a diagonal process, let us extract from the sequence $\{f_k\}_1^\infty$ a subsequence $\{f_{k_i}\}_{i=1}^\infty$ that converges to f almost everywhere in E^n. If we apply Fatou's lemma to the sequence $\{|f_{k_i} - f_{k_j}||g|\}$ (for $k_j = k_i, k_{i+1}, \ldots$), where $\|g\|_{p'} = 1$, and then use equation (4), we obtain

$$\int_{E^n} |f_{k_i} - f||g|\,dx \leqslant \sup_{k_s \geqslant k_i} \int_{E^n} |f_{k_i} - f_{k_j}||g|\,dx$$

$$\leqslant \sup_{k_j \geqslant k_i} \|f_{k_i} - f_{k_j}\|_p .$$

Since this inequality holds for arbitrary $g \in L_{p'}$ such that $\|g\|_{p'} = 1$, it follows that $(f - f_{k_i}) \in L_p$. This in turn implies (by Minkowski's inequality) that $f = (f - f_{k_i}) + f_{k_i} \in L_p$. It then follows from the last inequality on the basis of (3) that $\|f_k - f\|_p \to 0$ as $k \to \infty$. This proves property 4).

In what follows, we shall identify two equivalent functions (that is, two functions that coincide almost everywhere on G and hence have the same norm in the sense of $L_p(G)$), treating them as a single element of the space $L_p(G)$.

REMARK. Let f denote a member of L_p, so that $\|f\|_p < \infty$, and suppose that $p = (\overline{p}, \overline{\overline{p}})$, where $\overline{p} = (p_1, \ldots, p_j)$ and $\overline{\overline{p}} = (p_{j+1}, \ldots, p_n)$ for $1 \leqslant j \leqslant n - 1$. Then, the norm $\|f(\cdot, x)\|_{\overline{p}}$, as a function of the point $\overline{\overline{x}} = (x_{j+1}, \ldots, x_n)$, is measurable and almost everywhere finite. It is easy to show this successively for $j = 1, 2, \ldots, n - 1$ by using Fubini's theorem. For p_j representing infinite components of the vector p, we also use the relation (see, for example, Nikol'skiĭ [9], 1.1)

$$\lim_{q \to \infty} \left(\int_a^b |\varphi(t)|^q \, dt \right)^{1/q} = \operatorname{ess\,sup} |\varphi| \quad (b - a < \infty) \tag{5}$$

and the fact that the pointwise limit of measurable functions is a measurable function.