2 COLLECTED PAPERS OF

Kenneth J. Arrow

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Preface

The mutual connection, direct and indirect, of all parts of the economy is recognized in the classical economics of Adam Smith, Ricardo, and especially John Stuart Mill, though their analytic methods made the interrelations very simple indeed. A full explicit statement of general equilibrium was first achieved by Léon Walras, who stated clearly many of the leading problems, though with unsatisfactory resolutions: the existence of equilibrium, the efficiency of competitive equilibrium, its stability, its possible nonuniqueness, and the laws of its variation with respect to various parameters.

The subject, however, for all its intellectual grandeur, remained sterile. Little progress was made for many years, except for some insights of Pareto, who certainly clarified the relevant meaning of efficiency. The history is discussed more fully in Chapter 6; suffice it to say here that many of the basic issues were raised in the German literature of the 1930s and that the contributions of Abraham Wald and John von Neumann were basic to subsequent developments, to which Gerard Debreu, Leonid Hurwicz, and Lionel McKenzie were such major contributors.

The logic of general competitive equilibrium is closely related to that of economic planning. Leonid Hurwicz and I collaborated on a number of papers stimulated by this subject and leading to such questions as dynamic processes for achieving a maximum and the stability of competitive equilibrium. Our joint papers, together with other papers by each of us, have been collected in *Studies in Resource Allocation Processes* (1977), and have not been reprinted here. Chapter 12 here is a sample of work in this area.

The efficiency of competitive equilibrium holds under appropriate circumstances, but not under others, especially, as is well known, when there are externalities. In some papers, especially Chapter 7, I have sought to present the inefficiencies of the competitive system in a systematic form.

The papers that follow are among those I have published in technical journals, or as chapters in various types of collections, or as separate pamphlets. (Portions of books of which I was primary author are not included.) I am grateful to the publishers for permission to reproduce them here. The papers have been edited lightly, and a few have been supplied with headnotes to give the reader some insight into the circumstances that motivated the writing.

I would like to thank Mary Ellen Geer for her careful and thorough editing of this volume, Camille Smith for shepherding it through the publication process, and Michael Barclay and Robert Wood for preparing the index.

Contents

1 Alternative Proof of the Substitution Theorem for Leontief Models in the General Case

It is of some interest to state and prove, in a manner which does not involve the use of the calculus, the theorem concerning substitutability in Leontief models stated elsewhere by Samuelson (1951). The chief virtue of such restatement is not the generalization to nondifferentiable production functions but the greater clarity given to the importance of the special conditions of the problem. This approach has been developed by Koopmans (1951, chap. 8) for the case of three outputs; the present chapter seeks to generalize his results.

The Assumptions of the Samuelson-Leontief Model

Samuelson's assumptions will be restated here in the terminology of linear programming. We shall let $n + 1$ be the total number of commodities involved; the first n will be termed "products" and the $(n + 1)$th "labor."

ASSUMPTION 1. *There is a collection of basic activities, each represented by a vector with $n + 1$ components, such that every possible state of production is represented by a linear combination of a finite number of the basic activities with nonnegative coefficients.*[1] *The collection of basic activities from which such combinations are formed need not itself be finite.*

1. The restriction to linear combinations of a *finite* number of basic activities is unnecessary. The generalization of a set of nonnegative weights is a *measure* over the space of basic activities.

Reprinted from *Activity Analysis of Production and Allocation*, ed. T. C. Koopmans (New York: Wiley, 1951), pp. 155–164.

ASSUMPTION 2. *No basic activity has more than one output.*

ASSUMPTION 3. *In every basic activity labor is a nonzero input.*

ASSUMPTION 4. *There is a given supply of labor from outside the system, but none of any product.*

Assumption 1 is that of constant returns to scale; 2 states the absence of joint production in the basic activities; 3 states that labor appears solely as a primary input; and 4 states that no product is a primary input.

In the vector representation of activities, let the $(n + 1)$th component be labor. As usual, inputs will be represented by negative numbers, outputs by positive ones. By an activity of the ith industry we shall understand an activity in which no component other than the ith is positive. Clearly, any linear combination of the basic activities of the ith industry with nonnegative coefficients is itself an activity of the ith industry. Further, let y be any activity. Then, by Assumption 1,

$$y = \sum_k x_k b^k, \tag{1-1}$$

where $x_k \geqq 0$ and b^k is a vector representing a basic activity.[2] Number the activities b^k in such a way that those with $k = 1, \ldots, n_1$ are basic activities of the first industry, and, in general, those with $k = n_{i-1} + 1, \ldots, n_i$ are basic activities of the ith industry, where $n_0 = 0$. Then, from (1-1),

$$y = \sum_{i=1}^{n} \sum_{k=n_{i-1}+1}^{n_i} x_k b^k. \tag{1-2}$$

As noted,

$$\sum_{k=n_{i-1}+1}^{n_i} x_k b^k$$

(For the definition of a measure, see Saks, 1937, pp. 7–17.) If b stands for a variable basic activity and μ is a measure over the space of basic activities, then any state of production is of the form $\int b \, d\mu$. All subsequent results apply equally well to this more general case, with completely analogous proofs.

2. In this chapter all vectors are column vectors. For future reference, note that the prime symbol will not denote transposition but will serve to distinguish different column vectors.

is an activity of the ith industry. Hence, every activity is expressible as a sum of n activities, one from each industry.

Further, let a normalized activity be one in which the labor input is 1. From Assumption 1 it follows that every activity of the ith industry is the *nonnegative multiple* of a normalized activity of that industry, and conversely. Hence every activity is a linear combination of n normalized activities, one from each industry, with nonnegative coefficients. The amount of labor used in any activity is therefore the sum of these coefficients. If, finally, we choose the units of labor so that the total supply of labor available, as guaranteed by Assumption 4, is 1, we may say that any activity y is expressible in the form

$$y = \sum_{j=1}^{n} x_j a^j, \tag{1-3}$$

where

$$x_j \geqq 0, \quad \sum_{j=1}^{n} x_j = 1 \tag{1-4}$$

and a^j is a normalized activity of the jth industry. As now defined, all vectors y, a^j have -1 as their $(n + 1)$th component; let us redefine them to have *only their first n components.*

Note that the set of all normalized activities of the jth industry is a convex set; call it S_j. From Assumption 2,

$$\text{if} \quad a \in S_j, \quad \text{then} \quad a_k \leqq 0 \quad \text{for all } k \neq j. \tag{1-5}$$

(Here the symbol $\in$ means "belongs to"; a_k denotes the kth component of a.) Finally, it follows from Assumption 4 that

$$y \geqq 0. \tag{1-6}$$

(Following Koopmans [1951, chap. 3, sec. 2.5], we use this notation for partial ordering relations among vectors: $x \geqq y$ means $x_i \geqq y_i$, all i; $x \geq y$ means $x \geqq y$, $x \neq y$; $x > y$ means $x_i > y_i$, all i.)

The set S of feasible points in the product space is that satisfying (1-3), (1-4), and (1-6). The problem is to characterize the set of efficient points of S if the assumption contained in (1-5) is made.[3]

3. For the relevant definition of an efficient point see Koopmans (1951, chap. 3, sec. 5.2), considering labor as the only primary commodity.

The set of all points satisfying (1-3) and (1-4) will be referred to as the *convex hull of the union* of $S_1, \ldots, S_n$. S is then the intersection of the nonnegative orthant (of Euclidean n-space) with the convex hull of the union of $S_1, \ldots, S_n$.

The following notation and terminology will be used: A will denote a square matrix of order n, a_i^j will be the element in the ith row and jth column of A, and a^j will be the vector which is the jth column of A. A will be said to be *admissible* if $a^j \in S_j$ for every j. A *weight vector,* x, has the properties $x \geqq 0$, $\Sigma_{j=1}^n x_j = 1$. A pair (A,x) is said to be a *representation* if A is admissible, x is a weight vector, and $Ax \geqq 0$. A vector y for which there exists a representation (A,x) such that $y = Ax$ is termed *feasible;* this definition agrees with that given in the first sentence of the preceding paragraph.

In the light of (1-3)–(1-6), the economic significance of these definitions is obvious. In particular, the set of feasible points, or vectors, is precisely S; a representation is a mode of industrial organization which will achieve a given feasible point. Note that, in view of (1-5), $a_i^j \leqq 0$ for all $i \neq j$.

Two forms of Samuelson's theorem will be established, corresponding to Koopmans' "strong" and "weak" assumptions, respectively (1951, sec. 3.6). In the first case we assume that it is possible to produce a positive net output of all products; in the second we assume only that some net production is possible.

The Substitution Theorem under Strong Assumptions

THEOREM 1. *For each $j = 1, \ldots, n$, let S_j be a convex set in Euclidean n-space such that, if $a \in S_j$, then $a_i \leqq 0$ for $i \neq j$. Let S be the intersection of the convex hull of the union of $S_1, \ldots, S_n$ with the nonnegative orthant. If S is a compact set*[4] *with at least one positive element, then the set of efficient points of S is the intersection with the nonnegative orthant of an $(n-1)$-dimensional hyperplane the direction coefficients of whose outward normal are all positive.*

LEMMA 1. *If y' belongs to the compact set S, there is an efficient point y'' of S such that $y'' \geqq y'$.*[5]

Proof. Let U be the set of points y such that $y \in S$, $y \geqq y'$. U is a compact

4. That is, closed and bounded.

5. This lemma has been proved by von Neumann and Morgenstern (1947, p. 593) for the case where S has a finite number of elements.

set, so that the continuous function $\Sigma_{i=1}^{n} y_i$ attains a maximum in U, say at y''. Since $y'' \in U$, $y'' \geqq y'$. If y'' were not efficient, there would be a point $\bar{y}$ of S such that $\bar{y} \geq y''$; but then $\bar{y} \in U$, $\Sigma_{i=1}^{n} \bar{y}_i > \Sigma_{i=1}^{n} y_i''$, contrary to the construction of y''.

LEMMA 2. *If A is a (square) matrix such that $a_i^j \leqq 0$ for $i \neq j$, and x and y are vectors such that $Ax = y$, $x \geqq 0$, $y \geqq 0$, $y_i > 0$, then $x_i > 0$.*

Proof. By hypothesis, $a_i^j x_j \leqq 0$ for $i \neq j$, so that $\Sigma_{j \neq i} a_i^j x_j \leqq 0$. Hence $0 < y_i = a_i^i x_i + \Sigma_{j \neq i} a_i^j x_j \leqq a_i^i x_i$. Since $x_i \geqq 0$, we must have $x_i > 0$.

LEMMA 3. *Let A be a matrix such that $a_i^j \leqq 0$ for $i \neq j$ and for which there exists a vector x such that $Ax > 0$. Then (a) $Ax' \geqq 0$ implies $x' \geqq 0$; (b) $Ax' \geq 0$ implies $x' \geq 0$.*[6]

Proof. By Lemma 2, the hypothesis $Ax > 0$ implies $x > 0$. The ratios x_j'/x_j are therefore defined; let

$$m = \min_j (x_j'/x_j), \tag{1-7}$$

where j varies from 1 to n, and choose i so that

$$x_i'/x_i = m. \tag{1-8}$$

From (1-7) and the hypotheses,

$$a_i^j x_j' = a_i^j x_j (x_j'/x_j) \leqq a_i^j x_j m \qquad (j \neq i). \tag{1-9}$$

Suppose $Ax' \geqq 0$. Then, from (1-8) and (1-9),

$$0 \leqq \sum_{j=1}^{n} a_i^j x_j' \leqq a_i^i x_i m + \sum_{j \neq i} a_i^j x_j m = m \sum_{j=1}^{n} a_i^j x_j.$$

By hypothesis, $\Sigma_{j=1}^{n} a_i^j x_j > 0$, so that $m \geqq 0$. From (1-7), $x_j' \geqq 0$, since $x_j > 0$ for all j, establishing (a).

If $Ax' \geq 0$, then, clearly, $x' \geqq 0$ by (a), $x' \neq 0$, so that $x' \geq 0$.

LEMMA 4. *If A is a matrix such that $Ax \geqq 0$ only if $x \geqq 0$, then A is nonsingular.*

Proof. If x is such that $Ax = 0$, then $A(-x) = 0$. By hypothesis, $x \geqq 0$, $-x \geqq 0$, so that $x = 0$. Hence $Ax = 0$ implies $x = 0$, so that A must be nonsingular.

6. Recall that in this chapter the prime is not used as a transposition sign.

LEMMA 5. *If (A,x) is a representation of $y > 0$, let Q be the set of points $y' \geqq 0$ for which there exists a vector x' such that $Ax' = y'$, $\Sigma_{j=1}^{n} x_j' = 1$. Then every point of Q is feasible.*

Proof. By hypothesis,

$$(1\text{-}10) \qquad a_i^j \leqq 0 \quad \text{for} \quad i \neq j,$$

$$(1\text{-}11) \qquad Ax = y > 0.$$

From (1-10), (1-11), and Lemma 3, $Ax' \geq 0$ implies $x' \geq 0$. Since $\Sigma_{j=1}^{n} x_j' = 1$, x' is a weight vector. Therefore (A,x') is a representation, and y' is a feasible point.

LEMMA 6. *If Q is defined as in Lemma 5, there do not exist two points y', y'', in Q such that $y' \geq y''$.*

Proof. Suppose the contrary. Let $y' = Ax'$, $y'' = Ax''$, where

$$(1\text{-}12) \qquad \sum_{j=1}^{n} x_j' = 1 = \sum_{j=1}^{n} x_j'',$$

$A(x' - x'') \geq 0$. By the proof of Lemma 5, A satisfies the hypotheses of Lemma 3, so that $x' - x'' \geq 0$; but then, $\Sigma_{j=1}^{n}(x_j' - x_j'') > 0$, contrary to (1-12).

LEMMA 7. *If, for each $k = 1, \ldots, p$, $y^{(k)}$ has representation $(A^{(k)}, x^{(k)})$, and $t_k > 0$, and if $\Sigma_{k=1}^{p} t_k = 1$, then $y = \Sigma_{k=1}^{p} t_k y^{(k)}$ is feasible and has a representation (A,x), where $x = \Sigma_{k=1}^{p} t_k x^{(k)}$, and $a^j = (\Sigma_{k=1}^{p} t_k x_j^{(k)} a^{(k)j})/x_j$, for all j for which $x_j > 0$.*

Proof. Define x and a^j as in the hypothesis; for all j such that $x_j = 0$, choose a^j to be any element of S_j. Since the sets S_j are convex, it follows that $a^j \in S_j$ for each j, so that A is admissible. It is also easy to see that x is a weight vector, that $y = Ax$, and that $y \geq 0$, so that (A,x) is a representation of y.

LEMMA 8. *Let $y > 0$ be an efficient point with representation (A,x), and let T be defined in terms of y in the same way that Q is defined in Lemma 5. Then, (a) A is nonsingular; (b) every efficient point of S belongs to T.*

Proof. By the proof of Lemma 5, A satisfies the hypotheses of Lemma 3 and hence is nonsingular by Lemmas 3 and 4.

Let y' be any efficient point. Since there is a positive efficient point, we cannot have $y' = 0$. Since A is nonsingular, there is a vector x' such that

$Ax' = y' \geq 0$. By Lemma 3, $x' \geq 0$, and therefore $\Sigma_{j=1}^{n} x_j' > 0$. Let $t_0 = 1/\Sigma_{j=1}^{n} x_j'$. Then, $A(t_0 x') = t_0 y' \geq 0$, $\Sigma_{j=1}^{n} t_0 x_j' = 1$, so that

(1-13) $\quad t_0 y' \in T.$

By (1-13) and Lemma 5, $t_0 y'$ is feasible. If $t_0 > 1$, then $t_0 y' \geq y'$, which is impossible for an efficient point y'. Hence

(1-14) $\quad 0 < t_0 \leqq 1.$

The variable point $tt_0 y' + (1 - t)y > 0$ for $t = 0$. Hence we can choose t_1 so that

(1-15) $\quad t_1 < 0,$

(1-16) $\quad y'' = t_1 t_0 y' + (1 - t_1)y > 0.$

Let $x'' = t_1 t_0 x' + (1 - t_1)x$; then, by the definition of t_0 and the fact that x is a weight vector, $\Sigma_{j=1}^{n} x_j'' = 1$; also, $y'' = Ax''$. From (1-16) and the definition of T, $y'' \in T$. By Lemma 5,

(1-17) $\quad y''$ is a feasible point.

Let $t_2 = (t_1 t_0)/(t_1 t_0 - 1)$, $t_3 = (1 - t_1)/(1 - t_1 t_0)$. From (1-14) and (1-15),

(1-18) $\quad 0 < t_2 < 1,$

(1-19) $\quad t_3 \geqq 1.$

From (1-16),

(1-20) $\quad t_3 y = t_2 y' + (1 - t_2)y''.$

From (1-18), (1-20), (1-17), and Lemma 7, $t_3 y$ is a feasible point. If $t_3 > 1$, then $t_3 y > y$, so that y would not be efficient, contrary to hypothesis. Hence, from (1-19), $t_3 = 1$, which implies that $t_0 = 1$. From (1-13), then, $y' \in T$.

Proof of Theorem 1. By hypothesis, there is at least one positive feasible point. By Lemma 1, there is an efficient point $y > 0$. Let T be defined as in Lemma 8. Then every efficient point of S belongs to T. Conversely, let y' be any point of T. If y' is not efficient, there is, by Lemma 1, an efficient point $y'' \geq y'$. Since y'' is efficient, it belongs to T by Lemma 8; but this contradicts Lemma 6. Hence y' is efficient, so that T is precisely the set of efficient points.

T is the intersection with the nonnegative orthant of the hyperplane defined parametrically by the equations $Ax' = y$, $\Sigma_{j=1}^{n} x_j' = 1$. By Lemma 8,

A is nonsingular, so that $x' = A^{-1}y$. Let A^i_j be the element in the jth row and ith column of A^{-1}, and A^i be the ith column. Then the equation of the hyperplane is

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{n} A^i_j\right) y_i = 1.$$

Hence the numbers $\Sigma_{j=1}^{n} A^i_j$ are the direction numbers of the outward normal to T. For each i, AA^i is a vector all of whose components are zero except for the ith, which is 1. Therefore $AA^i \geq 0$; by Lemma 3, $A^i \geq 0$, so that $\Sigma_{j=1}^{n} A^i_j > 0$ for all i.

The Substitution Theorem under Weak Assumptions

A generalization of Theorem 1 in which it is assumed only that there is a feasible point $y \geq 0$ (instead of $y > 0$) will be developed in this section. Some new terminology and notation will be needed.

A representation (A,x) will be said to be *trivial* if there is a nonnull set of integers, I, such that $x_i > 0$ for some i in I, and $\Sigma_{j\epsilon I} a^j_i x_j = 0$ for all i in I. The mode of industrial organization displayed by a trivial representation has the property that there is a collection of industries in which there is some net input of labor and possibly of other commodities and such that the output of any one industry in the group is completely absorbed by the other industries in the group. This group, then, is only a drain on the net resources of the nation. The main result of this section is that any industry which can be used in any system of industrial organization not of the degenerate type just described can yield a positive net output; therefore Samuelson's theorem applies.

LEMMA 9. *Let A be a matrix such that $a^j_i \leqq 0$ when $i \neq j$; x and y vectors such that $x \geqq 0$, $y \geqq 0$, $y = Ax$; I a set of integers (between 1 and n); and i an element of I. Then, (a) $\Sigma_{j\epsilon I} a^j_i x_j \geqq y_i \geqq 0$; (b) if $\Sigma_{j\epsilon I} a^j_i x_j = 0$, then $y_i = 0$, and $a^j_i = 0$ for all $j \epsilon -I$ such that $x_j > 0$. (By $-I$ is meant the set of integers between 1 and n not in I.)*

Proof. From the hypothesis,

$$a^j_i x_j \leqq 0 \quad \text{for} \quad i \neq j, \tag{1-21}$$

so that

$$\sum_{j\epsilon -I} a^j_i x_j \leqq 0. \tag{1-22}$$

From (1-22) and the hypotheses,

$$0 \leqq y_i = \sum_{j\epsilon I} a_i^j x_j + \sum_{j\epsilon -I} a_i^j x_j \leqq \sum_{j\epsilon I} a_i^j x_j,$$

establishing (a). If $\Sigma_{j\epsilon I} a_i^j x_j = 0$, then clearly $y_i = 0$, and $\Sigma_{j\epsilon -I} a_i^j x_j = 0$, so that, from (1-21), $a_i^j x_j = 0$ for $j \epsilon -I$, from which (b) follows.

Lemma 9 is a generalization of Lemma 2.

LEMMA 10. *If $y \geq 0$ has a trivial representation (A,x), then y is not efficient.*

Proof. By hypothesis, there is a set of integers, I, such that

(1-23) $\quad x_i > 0 \quad$ for some $\quad i \epsilon I,$

(1-24) $\quad \sum_{j\epsilon I} a_i^j x_j = 0 \quad$ for all $\quad i \epsilon I.$

From (1-24) and Lemma 9b, $y_i = 0$ for all i in I; since $y_k > 0$ for some k, we must have k in $-I$. By Lemma 2, then, $x_k > 0$ for some k not in I. Together with (1-4), this shows that $0 < \Sigma_{j\epsilon I} x_j < 1$. Let $t = 1/(1 - \Sigma_{j\epsilon I} x_j)$, and define $x_j' = 0$ for $j \epsilon I$, $x_j' = tx_j$ for $j \epsilon -I$. Then

(1-25) $\quad t > 1,$

(1-26) $\quad x'$ is a weight vector.

Let $y' = Ax'$. For i in I, it follows from (1-24) and Lemma 9b that $a_i^j x_j = 0$ for j in $-I$. Hence

(1-27) $$y_i' = \sum_{j\epsilon I} a_i^j x_j' + \sum_{j\epsilon -I} a_i^j x_j' = 0 = ty_i,$$

for i in I. For $i \epsilon -I$, $a_i^j x_j \leqq 0$ for j in I. Therefore

$$0 \leqq y_i = \sum_{j\epsilon I} a_i^j x_j + \sum_{j\epsilon -I} a_i^j x_j \leqq \sum_{j\epsilon -I} a_i^j x_j,$$

so that

$$y_i' = \sum_{j\epsilon I} a_i^j x_j' + \sum_{j\epsilon -I} a_i^j x_j' = t \sum_{j\epsilon -I} a_i^j x_j \geqq ty_i,$$

for i in $-I$, or, with (1-27),

(1-28) $\quad y' \geqq ty.$

A is an admissible matrix by hypothesis; x' is a weight vector, by (1-26); and from (1-28), (1-25), and the hypothesis, $y' \geqq 0$, so that y' is a feasible point.

Furthermore, from (1-28), (1-25), and the hypothesis that $y \geq 0$, it follows that $y' \geq y$, so that y is not efficient.

The proof of Lemma 10 amounts to saying that the industrial organization represented by a trivial representation can always be improved by shutting down the group of industries which yields no net aggregate output and distributing the released labor to the other industries in proportion to the numbers already employed.

We shall also need the following generalization of Lemma 3:

LEMMA 11. *Let A be a matrix such that $a_i^j \leqq 0$ for $i \neq j$ and for which there exists a vector $x > 0$ such that (A,x) is a nontrivial representation. Then (a) $Ax' \geqq 0$ implies $x' \geqq 0$; and (b) $Ax' \geq 0$ implies $x' \geq 0$.*

Proof. Since $x > 0$, the ratios x_j'/x_j are defined. Let

$$m = \min_j (x_j'/x_j), \tag{1-29}$$

and let I be the set of integers such that $x_j'/x_j = m$; I is nonnull. From the hypothesis, $a_i^j x_j < 0$ for i in I, j in $-I$, if $a_i^j \neq 0$. We then have

$$x_j'/x_j = m \quad \text{for} \quad j \in I, \tag{1-30}$$

$$a_i^j x_j' = a_i^j x_j (x_j'/x_j) < m a_i^j x_j, \tag{1-31}$$

if i is in I, j in $-I$, and $a_i^j \neq 0$. Suppose that for all i in I, $\Sigma_{j \in I} a_i^j x_j = 0$; since $x_j > 0$ for all j, it would follow that (A,x) is trivial, contrary to hypothesis. Hence, by Lemma 9a, there is some i in I such that

$$\sum_{j \in I} a_i^j x_j > 0. \tag{1-32}$$

From (1-31),

$$\sum_{j \in -I} a_i^j x_j' < m \sum_{j \in -I} a_i^j x_j, \tag{1-33}$$

if $a_i^j \neq 0$ for some j in $-I$. Suppose $Ax' \geqq 0$. Then, using (1-30),

$$0 \leqq \sum_{j \in I} a_i^j x_j' + \sum_{j \in -I} a_i^j x_j' = m \sum_{j \in I} a_i^j x_j + \sum_{j \in -I} a_i^j x_j'. \tag{1-34}$$

If $a_i^j = 0$ for all j in $-I$, then, from (1-32) and (1-34), it follows that $m \geqq 0$. If $a_i^j \neq 0$ for some j in $-I$, then, from (1-33) and (1-34),

$$0 < m \sum_{j=1}^{n} a_i^j x_j.$$

Since $\Sigma_{j=1}^{n} a_i^j x_j \geqq 0$ by the hypothesis that (A,x) is a representation, we must have $m > 0$. Hence, in either case, it follows from (1-29) that $x' \geqq 0$. Part (b) follows from (a) as in Lemma 3.

An integer, i, between 1 and n will be said to denote a *useful industry* if there is some nontrivial representation (A,x) in which $x_i > 0$. Lemma 10 guarantees us that, in the search for efficient points, industries which are not useful can be regarded as nonexistent, so there is no loss of generality in assuming that all numbers denote useful industries.

It is possible that the set of feasible points is empty, in which case Samuelson's theorem naturally has no particular content. Hence we shall assume that there is at least one useful industry.

THEOREM 2. *For each $j = 1, \ldots, n$, let S_j be a convex set in Euclidean n-space such that if $a \in S_j$, then $a_i \leqq 0$ for $i \neq j$. Let S be the intersection of the nonnegative orthant with the convex hull of the union of $S_1, \ldots, S_n$. If S is a compact set, and if every number from 1 to n denotes a useful industry, then the set of efficient points of S is the intersection with the nonnegative orthant of a hyperplane the direction coefficients of whose outward normal are all positive.*

Proof. For each k, let $y^{(k)}$ be a feasible point with a nontrivial representation $(A^{(k)},x^{(k)})$ such that $x_k^{(k)} > 0$ for each k; the existence of these points follows from the hypothesis that every number from 1 to n denotes a useful industry. Let $y = (\Sigma_{k=1}^{n} y^{(k)})/n$; by Lemma 7, y is a feasible point with representation (A,x), where $x = (\Sigma_{k=1}^{n} x^{(k)})/n$, so that $x > 0$, and $a^j = (\Sigma_{k=1}^{n} x_j^{(k)} a^{(k)j})/nx_j$. Suppose (A,x) is trivial; then, for some set of integers I, $\Sigma_{j \in I} a_i^j x_j = 0$ for all i in I. From this, it follows that

$$\sum_{k=1}^{n} \left(\sum_{j \in I} a_i^{(k)j} x_j^{(k)} \right) = 0,$$

for all i in I. From Lemma 9a, then, $\Sigma_{j \in I} a_i^{(k)j} x_j^{(k)} = 0$ for each k and all i in I; in particular, the equation holds for any k in I. Since $x_k^{(k)} > 0$, and therefore $x_i^{(k)} > 0$ for at least one i in I, we would have $(A^{(k)}, x^{(k)})$, a trivial representation, contrary to hypothesis. Hence (A,x) is a nontrivial representation with $x > 0$. All the conditions of Lemma 11 are satisfied, so that, by Lemmas 11 and 4, A is nonsingular.

Let y' be any positive vector. Then there is a vector x' such that $Ax' = y' > 0$. By Lemma 11, $x' \geq 0$; let $t = 1/(\Sigma_{j=1}^{n} x_j') > 0$. Then tx' is a weight vector, and $ty' = A(tx')$ is a positive feasible point with representation