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Optimal Control of Partial Differential Equations II: Theory and Applications

Conference held at the Mathematisches Forschungsinstitut, Oberwolfach, May 18–24, 1986

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PREFACE

This volume contains the contributions of participants of the conference "Optimal Control of Partial Differential Equations" which, under the chairmanship of the editors, took place at the Mathematisches Forschungsinstitut Oberwolfach from May 18 to May 24, 1986.

The great variety of topics covered by the contributions strongly indicates that also in the future it will be impossible to develop a unifying control theory of partial differential equations. On the other hand, there is a strong tendency to treat problems which are directly connected to practical applications. So this volume contains real-world applications like optimal cooling laws for the production of rolled steel or concrete solutions for the problem of optimal shape design in mechanics and hydrodynamics. Another main topic is the construction of numerical methods. This includes applications of the finite element method as well as of Quasi-Newton-methods to constrained and unconstrained control problems. Also, very complex problems arising in the theory of free boundary value problems are treated. Finally, some contributions show how practical problems stimulate the further development of the theory; in particular, this is the case for fields like suboptimal control, necessary optimality conditions and sensitivity analysis.

As usual, the lectures and stimulating discussions took place in the pleasant atmosphere of the Mathematisches Forschungsinstitut Oberwolfach. Special thanks of the participants are returned to the Director as well as to the staff of the institute.

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ON A DOMAIN OPTIMIZATION PROBLEM IN HYDROMECHANICS

T.S. Angell and R.E. Kleinman

Abstract. Certain hydromechanical quantities associated with a floating or a totally immersed body depend explicitly on the body's geometry. We discuss some aspects of the mathematical description of such a physical system and consider the problem of choosing the shape of the body so that one such quantity is optimized. Certain families of solutions of the original boundary value problem are shown to be complete and a penalization method for treating the optimization problem is proposed.

1. INTRODUCTION

When a body, floating on the surface of an infinite, ideal, inviscid, irrotational fluid is subjected to a periodic vertical displacement, a wave pattern is created in the fluid. The problem of determining this pattern from a knowledge of the body geometry and applied forces is well known in fluid mechanics.

In problems with both partially and fully submerged objects quantities of physical interest are not only the wave patterns which may be derived from the velocity potential, but also functionals of the potential such as added mass and damping factors which measure the distribution of energy in the fluid, e.g. Weyhausen and Laitone [24 , p.567]. These factors are, of course, dependent on the body geometry and the natural question arises as to whether such quantities may be optimized over restricted classes of body geometry.

The question of optimizing the added mass (and similar) functionals was addressed by Angell, Hsiao, and Kleinman [2] who studied the problem for a body which is totally submerged in a fluid of finite depth. In the terminology of optimal control the problem is one of optimization of geometrical elements (see e.g. J.L. Lions [17]). Other optimization problems of this general class have been studied previously by, for example, Cea and his coworkers [7], [8], Chenais [9], and Pironneau [21], [22]. However, in contrast to much of the earlier work, the natural setting of our problem is in an unbounded rather than in a bounded domain.

It will come as no surprise to those familiar with the peculiar difficulties associated with exterior boundary value problems that it is particularly useful to reformulate the original boundary value problem (which here includes not only boundary conditions given on the bounded surface of the body, but also on the free surface and on the bottom both of which are of infinite extent) as a uniquely solvable integral equation defined on the boundary of the body.

Before summarizing the contents of the remaining sections of this paper, it is perhaps worthwhile to pause and make some comment on the boundary value problem itself and on its reformulation in terms of boundary integral equations. Unlike some other areas of mathematical physics as, for example the linearized theories of acoustics or electromagnetics, there are here certain basic questions to which we have only partial answers.

In his classic paper [13], F. John formulated the problem of a partially immersed heaving body as a boundary value problem for the velocity potential which satisfies Laplace's equation with given Neumann data on the submerged portion of the boundary, a linearized free surface condition on the mean free surface (the fluid-air boundary), a homogeneous Neumann condition on the bottom of the fluid, and a radiation condition. Successful application of the boundary integral equation method usually depends on knowing that the boundary value problem can have at most one solution, and it is just at this point that interesting problems appear. He established uniqueness only with restrictions on the body shape, in particular that it be convex, smooth and have normal intersection with the free surface and, moreover, that vertical rays from the free surface intersect the body at most once. Certainly, these restrictions are required by the technique of proof. To quote John: "There appears to be no physical reason why in [the contrary] cases the primary wave motion together with the motion of the obstacle should not determine the motion of the liquid uniquely." [13, p.49].

Since the appearance of John's paper in 1950 others have returned to these and other questions raised by John's analysis. Thus, for example, Kleinman has shown in [15] that the geometrical conditions may be somewhat relaxed: corners are allowable as are non-normal intersections with the free surface, and convexity is not necessary. Concerning the requirement that vertical rays intersect the body only once, the condition has recently been relaxed in the two dimensional case by Simon and Ursell [23] although even their work does not constitute a general proof of uniqueness for all configurations. We will not give a complete review here of the work on the uniqueness question since the first part of this latter paper reviews the current knowledge on that topic.

When the body is completely submerged, John's uniqueness proof no longer applies. However Maz'ja [19] has provided a proof for a class of bodies delimited, once again, by certain geometric restrictions. The recent and interesting paper of A. Hulme [12] discusses the result of Maz'ja and most effectively describes the geometric meaning of the result. We will give a precise statement of this result in the next section. At this point, suffice it to say that the condition of Maz'ja provides a reasonable class of bodies for which, in the case that the body is totally submerged, we can assert the uniqueness of solutions of the boundary value problem.

In the reformulation of the exterior boundary value problem using boundary integral equations, John employed the Green's function for the entire fluid domain with no body present that satisfied the boundary condition at the bottom of the fluid (assumed flat) and the linearized free-surface condition, on the entire fluid-air boundary. Moreover, he demonstrated the existence of "irregular frequencies" i.e. values of the coupling parameter which appears in the free-surface condition, for which the integral equation is not uniquely solvable. We emphasize that this unique solvability question does not concern the original boundary value problem but rather the integral equation: we assume unique solvability of the former, and our object is to discover how to avoid irregular frequencies in the latter.

Recently, Kleinman [15] provided two methods of modifying the integral equation so that there were no irregular frequencies. In one case, the domain of the integral operator was enlarged and in the other the operator itself changed but both methods employed John's Green's function which is rather complicated. L. Wienert [25] has addressed the question of solvability of a boundary integral equation derived for the same class of bodies with this Green's function.

Another way to treat this problem is to employ a much simpler Green's function, one that satisfies only the boundary condition at the bottom of the fluid. Since this function does not satisfy the free surface condition one obtains an integral equation defined over both the surface of the body and the free surface. Such an integral equation was derived and even solved numerically for certain cases, e.g. Yeung [26] and Bai and Yeung [6]. Angell, Hsiao and Kleinman [1] showed that this integral equation, in the three dimensional case, has no irregular frequencies. More recently Liu [18] has studied the two dimensional case, both theoretically and numerically.

In the case of the totally submerged body we were able to show in our earlier paper [2] that the integral equation arising from the use of John's Green's function is uniquely solvable for all frequencies. It is the formulation of the boundary value problem, the statement of Maz'ja's theorem and the derivation of this boundary integral equation which form the content of the next section, while Section 3 contains a description of the optimization problem and a statement of the results obtained in [2] concerning the existence of an optimal body shape.

In the final section, Section 4, we turn to the question of a constructive method for finding approximate optimal surfaces. We prove there that certain families of functions form complete families of solutions and propose a penalization-type method for the constructive solution. The idea of using complete families to find approximate solutions to elliptic equations goes back at least to the work of Picone and of Fichera (see Miranda [20] for references). Angell and Kleinman [3], [4] have used such families in treating some optimization problems which arise in acoustic and in electromagnetic radiation problems. An approximation method similar to that proposed here is discussed in the context of an inverse transmission problem by Angell, Kleinman and Roach [5]. A related method in the inverse acoustic problem has been reported by Kirsch and Kress [14].

2. THE EXTERIOR BOUNDARY VALUE PROBLEM

We are concerned with solutions of Laplace's equation in an unbounded domain D^+ in $\mathbb{R}^3$ exterior to a bounded boundary, Γ, which is assumed to be a Lyapunov surface of index 1. A Cartesian coordinate system is fixed with origin in the unbounded region determined by Γ in terms of which the domain $D^+ = \mathbb{R}^2 \times [-h,0]$ as indicated in the accompanying figure:

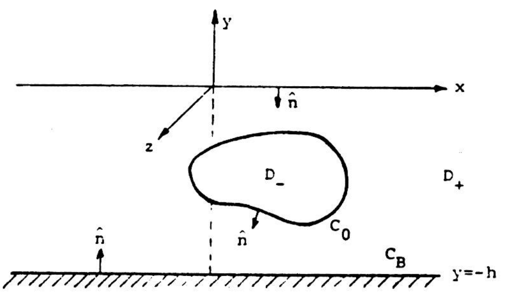

Fig.1

Indeed, the submerged body, whose interior we denote by D_-, will be assumed to be simply connected and lie in a strip $\mathbb{R}^2 \times [-h+\epsilon_0, \epsilon_0]$, $\epsilon_0 > 0$. The condition that the surface be Lyapunov of index 1 guarantees, among other things, that there exists an Lipschitz continuous normal $\hat{n}$ at all points of Γ. We emphasize that $\hat{n}$ is oriented so that it points into D_+. Points will be denoted by $p = (x_p, y_p, z_p)$ with cylindrical coordinates $p = (\rho_r, \theta_r, y_p)$ and the subscripts will be omitted if there is no danger of confusion.

With these conventions in mind, we will concern ourselves with the following boundary value problem:

$$
\begin{aligned}
&\text{(a)} \quad \Delta\phi = 0 \quad \text{in} \quad D_+ \,, \\
&\text{(b)} \quad \frac{\partial\phi}{\partial n} + k\phi = 0 \quad \text{on} \quad y = 0 \,, \\
&\text{(c)} \quad \frac{\partial\phi}{\partial n} = 0 \quad \text{on} \quad y = -h \,, \\
&\text{(d)} \quad \frac{\partial\phi}{\partial n} = g \quad \text{on} \quad \Gamma \,,
\end{aligned} \tag{2.1}
$$

together with a radiation condition

$$\text{(e)} \quad \frac{\partial\phi}{\partial\rho} - ik_o\phi = 0(\rho^{-\frac{1}{2}}) \,.$$

In this formulation, $g \in C(\Gamma)$ and k_o is the root with the largest real part of the transcendental equation

$$k_n \sinh k_n h = k \cosh k_n h \,. \tag{2.2}$$

In [19] Maz'ja introduced a restricted class of boundaries for which this boundary value problem has at most one solution. We formulate that theorem as follows.

<u>Theorem 2.1.</u> Let V be the vector field in $\mathbb{R}^3$ defined by

$$V = \left[\frac{\rho(y^2 - \rho^2)}{\rho^2 + y^2}\right] \tilde{\rho} - \left[\frac{2\rho^2 y}{\rho^2 + y^2}\right] \tilde{y} \,.$$

Then the homogeneous boundary value problem (1) with $g = 0$ has only the trivial solution provided

$$V \cdot \hat{n} \geq 0 \quad \text{on} \quad \Gamma \,. \tag{2.3}$$

A discussion of this result and its geometric significance may be found in A. Hulme [12]. We will refer to the class of all such surfaces as the Maz'ja class.

Following John [13] we introduce the Green's function for this problem which is normalized to have the form

$$\gamma(p,q) = -\frac{1}{2\pi}\frac{1}{|p-q|} + R(p,q) \tag{2.4}$$

where the function R has bounded derivatives with respect to q for points $p \in \Gamma$ (see[13,p.96] and γ satisfies conditions 2.1b,c and e. Using this Green's function to define single and double layer potentials, the usual jump conditions can be established as in the potential-theoretic case since the singular behavior of γ and $\frac{\partial}{\partial n_q}\gamma$ is determined by the first term in (2.4). For convenience, we record these results here:

$$\lim_{p\to\Gamma^\pm} \frac{\partial}{\partial n_p} \int_\Gamma u(q)\,\gamma(p,q)\,d\Gamma_q = \pm u(p) + \int_\Gamma u(p) \frac{\partial\gamma(p,q)}{\partial n_p}\, d\Gamma_q \,, \qquad (2.5)$$

$$\lim_{p\to\Gamma^\pm} \int_\Gamma u(q) \frac{\partial}{\partial n_q} \gamma(p,q)\,d\Gamma_q = + u(p) \mp \int_\Gamma u(q) \frac{\partial\gamma(p,q)}{\partial n_q}\, d\Gamma_q \qquad (2.6)$$

where $p \to \Gamma^\pm$ means p approaches Γ from $D^\pm$, $u \in L_2(\Gamma)$ and the relations (2.5) and (2.6) hold in the L_2 sense [20].

Moreover, if ϕ is a solution of the boundary value problem (2.1) then one may use Green's Theorem to establish the familiar relation

$$\int_\Gamma \left[\gamma(p,q)\frac{\partial \phi(q)}{\partial n_q} - \phi(q)\frac{\partial}{\partial n_q}\gamma(p,q)\right] d\Gamma_q = \begin{cases} 2\phi(p), & p \in D^+ \\ \phi(p), & p \in \Gamma \\ 0, & p \in D^- \end{cases} . \qquad (2.7)$$

If one then uses the boundary condition (2.1d) we have, for $p \in \Gamma$,

$$\int_\Gamma \gamma(p,q)\,g(q)\,d\Gamma_q - \int_\Gamma \phi(q)\frac{\partial}{\partial n_q}[\gamma(p,q)]\,d\Gamma_q = \phi(p) \qquad (2.8)$$

or, in operator notation

$$(I + \bar{K}^*)\,\phi = \int_\Gamma \gamma(p,q)\,g(q)\,d\Gamma_q \qquad (2.9)$$

where $\bar{K}^*$ is the boundary integral operator with kernel $\partial\gamma/\partial n_q$. We pause to remark that, given a solution, u, of this integral equation we may represent the solution of the boundary value problem according to the relation (2.7) by

$$\phi(p) = \frac{1}{2}\int_\Gamma \gamma(p,q)\,g(q)\,d\Gamma_q - \frac{1}{2}\int_\Gamma u(q)\frac{\partial\gamma(p,q)}{\partial n}\,d\Gamma_q \,, \quad p \in D^+ \qquad (2.10)$$

and, again using the jump relations, one sees easily that

$$\phi\big|_\Gamma = u \,, \qquad (2.11)$$

which is a direct relationship between the solution of the boundary integral equation and the boundary values taken on by the solution. Such a direct relation does not obtain when one uses a layer approach in which one assumes that the solution ϕ had a representation as a single layer,

$$\phi(p) = \int_\Gamma u(q)\,\gamma(p,q)\,d\Gamma_q \,,$$

and then uses the boundary condition and jump relations to obtain an integral equation for u.

As we will see below when we consider the optimization problem, it is particularly convenient to have this formulation as the cost functional involves just the trace of the solution ϕ of (2.1) on Γ .

With the aid of these jump conditions, we have proved the unique solvability of the boundary integral equation (2.9). Specifically, we may state the following theorem referring to [2] for the proof:

Theorem 2.2: Let Γ be Lyapunov of index 1 and belong to the Maz'ja class. Let $g \in C(\Gamma)$. Then the integral equation (2.9) has a unique solution in $L^2(\Gamma)$.

Remark: In fact, the solution whose existence is guaranteed by this last theorem can be shown, by a standard argument to be continuous since $g \in C(\Gamma)$. However, we will not require this result in what follows.

3. THE OPTIMIZATION PROBLEM

Let $\Gamma_0 = \{p \in \mathbb{R}^3 \,|\, |p| = 1\}$ denote the surface of the unit ball in $\mathbb{R}^3$ and let $C^{1,1}(\Gamma_0)$ denote the space of continuously differentiable functions whose first derivatives satisfy a Lipschitz condition and which is equipped with the usual Hölder norm $||\cdot||_{1,1}$ (see e.g.[10]). We will assume that we are given a family of surfaces which can be described by $C^{1,1}$ parameterizations:

$$\Gamma(f) = \left\{ p \in \mathbb{R}^3 \,|\, p = f(\tilde{p})\tilde{p} + p_0 \,,\ \tilde{p} = \frac{p - p_0}{|p - p_0|} \right\}$$

where $f: \Gamma_0 \to \mathbb{R}^3$ is an element of $C^{1,1}(\Gamma_0)$ and $p_0 \in \mathbb{R}^2 \times (-h+\epsilon_0, -\epsilon_0)$. Let a and b be two positive constants and define the subset $F_{a,b} \subset C^{1,1}(\Gamma_0)$ by

$$F_{a,b} = \{ f \in C^{1,1}(\Gamma_0) \,|\, ||f||_{1,1} \le b, f(\tilde{p})\tilde{p} + p_0 \in \mathbb{R}^2 \times (-h+\epsilon_0, -\epsilon_0) \tag{3.2}$$

and $\quad f(\tilde{p}) \ge a, \ \tilde{p} \in \Gamma_0\}$.

Definition 3.1. A surface S in $\mathbb{R}^3$ will be called admissible provided S can be described by a parametization $f \in F_{a,b}$ and S is contained in the Maz'ja class (c.f. Theorem 1.1).