PROCEEDINGS OF THE

THIRD INTERNATIONAL CONFERENCE ON FINITE ELEMENTS IN FLOW PROBLEMS

BANFF, ALBERTA, CANADA

10-13 JUNE, 1980

VOLUME I

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EDITOR: D. H. NORRIE

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SOME DEVELOPMENTS OF THE FINITE ELEMENT METHODS FOR FLUID MECHANICS

by

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ABSTRACT

The paper reviews and summarized the developments in:

(a) Penalty methods for enforcement of incompressiblity constraints.
(b) Galerkin-Petrov, upwind methods for concrete-diffusive problems.
(c) Hiererchial concepts of element formulation and introduces;
(d) new semi-explicit time stepping procedures.

INTRODUCTION

More than a decade has now elapsed since the entry of the finite element method into the arena of numerical fluid mechanics. Its unlimited potential as a unified approximation procedure is being realized today but many special developments are required to make its applications economically advantageous.

In the opening lectures of the first and second conferences of this series the first author had the priviledge of introducing two developments which today have become established and widely used. The first of these, in the 1974 conference, was the introduction of penalty formulation [1],[2] for simple treatment of incompressible flow. The second, in the 1976 conference was the Petrov-Galerkin treatment of optimally "upwinded" convective terms. [3],[4],[5] At this, third, conference we shall;

(a) survey the development of both 'upwinding' and penalty procedures;
(b) introduce a special penalty formulation not necessitating the use of reduced integration.
(c) draw attention to 'hierarchical' finite element concepts which have recently been introduced with success to the solid mechanics field and finally;
(d) introduce a class of new time stepping methods which promises to improve the economics of computation.

PENALTY PROCEDURES

Many physical processes are governed by differential equations of the type

$$\begin{cases} \underset{\sim}{L}\underset{\sim}{u} + \overline{\underset{\sim}{C}}^T p = \underset{\sim}{f} \\ \underset{\sim}{C}\underset{\sim}{u} = 0 \end{cases} \qquad (1)$$

where $\underset{\sim}{L}$, $\overline{\underset{\sim}{C}}$ and $\underset{\sim}{C}$ are differential operators and $\underset{\sim}{u}$, p the independent variables.

The Navier-Stokes equations of fluid mechanics are a particular case of above with $\underset{\sim}{u}$ standing for the velocities and p for the pressures. Further, in this particular case the operator

$$\underset{\sim}{C} = \overline{\underset{\sim}{C}} \equiv \underset{\sim}{\nabla}^T \qquad (2)$$

is a constraint imposing incompressibility.

In penalty formulation we first approximate equation (1) by writing

$$\underset{\sim}{L}\underset{\sim}{u} + \bar{\underset{\sim}{C}}^T p = \underset{\sim}{f}$$
$$\underset{\sim}{C}\underset{\sim}{u} - \frac{p}{\alpha} = 0 \qquad (2)$$

where, with $\alpha \to \infty$ the original formulation is obtained.

The device of introducing the penalty parameter, α , has the advantage of permitting the elimination of p, either before or after, discretization avoiding the difficulties associated numerically with typical, zero diagonals, of Lagrangian form.

The two alternate approaches lead to discretization which can be written either as

$$\begin{bmatrix} \underset{\sim}{K}_{uu}, & \underset{\sim}{K}_{up} \\ \underset{\sim}{K}_{pu}, & \frac{1}{\alpha} \underset{\sim}{K}_{pp} \end{bmatrix} \begin{Bmatrix} \bar{\underset{\sim}{u}} \\ \bar{\underset{\sim}{p}} \end{Bmatrix} = \begin{Bmatrix} \bar{\underset{\sim}{f}} \\ 0 \end{Bmatrix} \qquad (3)$$

with

$$\underset{\sim}{u} = \underset{\sim}{N}\bar{\underset{\sim}{u}}; \quad p = \bar{\underset{\sim}{N}}\bar{p}$$

Equation (3) leads on elimination of $\bar{\underset{\sim}{p}}$ to a form

$$\left(\underset{\sim}{K}_{uu} + \alpha\hat{\underset{\sim}{K}}\right) \bar{\underset{\sim}{u}} = \bar{\underset{\sim}{f}} \qquad (4)$$

If alternatively p is eliminated at the differential equation level and discretization adopted for $\underset{\sim}{u}$ alone we have a form

$$\left(\underset{\sim}{K}_{uu} + \alpha\hat{\hat{\underset{\sim}{K}}}\right)\bar{\underset{\sim}{u}} = \bar{\underset{\sim}{f}} \qquad (5)$$

where in general

$$\hat{K} \neq \hat{\hat{K}}$$

The evaluation of the matrix K in the first form of Eq. 4 is cumbersome and costly for C^0 continuous p discretization. However, if the weak form of Eq. 1 permits discontinuous approximations, the parameters $\bar{\underset{\sim}{p}}$ can be eliminated at element level. For particular local discretizations of this form it can be shown that formulation (4) became identical with those of (5) in which quadrature rule adopted for $\hat{\hat{K}}$ uses identical number of values as the number of parameters required for p determination.[6]

It will be realized that the number of parameters of $\bar{p}$ introduced (or the number of quadrature points used) is equivalent to the number of constraints of incompressibility imposed. If this number is 'm' and the number of parameters describing variation of u is 'n' then clearly for nontrivial solutions

$$n > m$$

is needed. In Fig. 1, we show the performance of various elements on the basis of a number

$$\beta = \frac{\Delta n}{\Delta m} \qquad (6)$$

for addition of a single element to a constrained field. Obviously $\beta < 1$ will lead to meaningless results.

It is readily seen why the serendipity type element originally used (1) has now been superceded by linear [7] and parabolic [8] Lagrangian elements with reduced/selective integration. Using in place of reduced integration - locally defined 3 parameter linear expansion allows us to resurect the serendipity element as an efficient non-locking one.[9][10]

Same comments apply to three dimensional problems where again an extremely efficient, serendipity type, brick with 20 nodes has been developed using four internal constraints.

OPTIMAL, UPWINDED PETROV-GALERKIN SCHEMES

The original concepts of 'upwinding' or the use of non-symmetric weighting functions have been suggested by some equivalent finite difference ideas but have not followed the same path. Thus many objections (of excessive diffusion or inaccuracy) have been laid to rest. It has been repeatedly shows that optimally upwinded schemes not only eliminate 'wiggles' [11], but at all stages produce more accurate solutions in the steady state convective-diffusive problems than do symmetric Galerkin Bubnov type operators (even in the extreme case when both numerical solutions are 'wiggle' free).

The interest in this area of finite element approximation has found its expression in a recent conference on this subject[12]. Despite much discussion it can be concluded that the original form of upwinding for one dimensional and two dimensional elements[3][5] is still not superceded although variants in the formulation using special quadrature rules [13] or equivalent compensating diffusion[14][15] ease numerical compu tion.

Extension of upwinding to the Navier-Stokes equations has followed a similar path and today this is widely used in that context[16-19].

For transient, non-steady state convective-diffusion problems much yet remains to be done but clearly if such problems tend to steady state precisely the same amount of 'upwinding' will be necessary. Thus concentration of activity on reduction of 'error' in time marching schemes is of primary importance.

HIERARCHICAL CONCEPTS AND ITERATIVE SOLUTIONS

As the finite element method is an approximation process it is frequently necessary to use two or more mesh refinements to obtain an accurate solution. In the standard use of finite elements the subdivided mesh does not contain the coarser mesh shape functions and in general no use is made of the coarse mesh solution to obtain the refined one. If on the other hand, shape functions are arranged 'hierarchically' in such a manner that the refined variables (a_j) are superposed on the coarse ones (a_i) then the matrix discretized forms are

for coarse mesh

$$\underset{\sim}{K}_{ii}\, \underset{\sim}{a}_i = \underset{\sim}{f}_i \tag{7}$$

for fine mesh

$$\begin{bmatrix} \underset{\sim}{K}_{ii}, & \underset{\sim}{K}_{ij} \\ \underset{\sim}{K}_{ji}, & {}_{jj} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{a}i \\ \underset{\sim}{a}j \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{f}_i \\ f_j \end{Bmatrix} \tag{8}$$

Here not only the original (coarse mesh) matrix reappears but $\underset{\sim}{a}i$, the coarse mesh solution forms a first approximation to the iterative solution of the refined variables.

Hierarchical funtions arise naturally in the 'global-local' approximations of Mote [20] or in successive addition of polynomial modes to isoparametric elements[21]. The realization of the power of the process for refinement of solution by iteration can be credited independently to Wilson [22], Wachspress[23],Peano[24][25], Szabo[26]. Indeed the latter two authors and their collaborators have extended the process to self-adaptive mesh refinement procedures [viz. Babuska 27][28].

While all the applications of the hierarchical concepts have so far been made in the context of linear solid mechanics, their use in non-linear fluid mechanics area is yet to come. If we consider the fact that iterative processes are invariably necessary in such problems and that these are the major part of the cost of the operation it seems that the use of hierarchical concepts is here overdue.

In the finite difference context parallel progress has been made using so called multi-grid procedures in which acceleration of iteration by simultaneous use of coarse and fine grids together with appropriate interpolation has proved successful.[28] Clearly in finite element form where the link between successive refinements is clearer the process would be even more advantageous.

A NEW SET OF TIME STEPPING ALGORITHM (SEE) SELECTIVE EXTRAPOLATED EXPLICIT PROCESSES

The solution of unsteady state fluid mechanics problems (or indeed the use of such usteady solutions as an interation device to obtain steady states) necessitates the use of efficient time stepping processes. Many such processes are now known and well documented; further most of these can be developed via the applicaiton of finite element concepts to the time domain.

With 'explicit' processes the computation of each time step is trivial as no equation solving processes are involved. Unfortunately such schemes are plaqued by instability and an extremely small time step is generally neede to overcome this. Unconditionally stable, implicit methods requires on the other hand a full equation solution to be carried out at each computational step and thus the cost per time step as very much larger. In the finite difference context a very much economical compromise has been achieved in the so called ADI (Alternating Direction Implicit) processes [31/33] and some attempts at grafting these methods on to the finite element formulations have recently been made.[33]

Here we shall outline general partioning/staggered processes which are similar to the ADI methods but which can in the limit be reduced to a fully explicit and unconditionally stable methods. We shall class the method under the name of SEE (Selective Extrapolated Explicit) Process .

The motivation for the methods here described comes from procedures developed for the solution of coupled physical systems by Parks and Fellipa [34],[35] the approach below assumes that we are dealing with two sets of (discretized) physical variables a_1 and a_2 which obey the coupled equations of first order

$$C_{11}\,\dot{a}_1 + K_{11}a_1 + \left[C_{12}\,\dot{a}_2 + K_{12}\,a_2\right] = f_1 \qquad (9a)$$

$$C_{22}\,\dot{a}_2 + K_{22}\,a_2 + \left[C_{21}a_1 + K_{12}\,a_2\right] = f_2 \qquad (9b)$$

Considering the above system we can proceed as follows. First, by the use of some integration algorithm, solve equation (9a) for the variable a_1^{n+1} using known values of a_1^n a_1^{n-1} etc., and extrapolated values of $\dot{\bar{a}}_2^x$ and $\dot{\bar{a}}_2^x$. Second, apply a similar solution to Eq. (9b) for a_2^{n+1} using now the values of a_1^{n+1} computed in the first step of the operation.

Using unconditionally stable although not necessarily identical algorithms, for each stage of the operation and a suitable extrapolation procedure unconditional stability of the whole process can be assured.

Consider for instance the application of linear expansion of a_1 and a_2 and the use of suitable weighting resulting in a two point θ , algorithm [36] to Eq. (9.1) We have now, at stage 1 of the operation

$$C_{11}\left(\underset{\sim}{a}_i^{n+1} - \underset{\sim}{a}^n\right) + \Delta t K_{11}\left(\theta_1 \underset{\sim}{a}_1^{n+1} + (1-\theta_1)\,\underset{\sim}{a}_1^n\right) = \qquad (10)$$
$$\Delta t \bar{f}_1 - C_{12}\,\dot{\bar{\underset{\sim}{a}}}_2^x - \Delta t K_{12}\,\bar{\underset{\sim}{a}}_2^x$$

where the extrapolation of the mean values $\dot{\bar{\mathbf{a}}}_2^{*}$ and $\bar{\mathbf{a}}_2^{*}$ can proceed in a variety of ways e.g. writing

$$\dot{\bar{\mathbf{a}}}_2^{x} = \beta(\mathbf{a}_2^{n} - \mathbf{a}_2^{n-1})$$

$$\bar{\mathbf{a}}_2^{x} = \theta_1 \left(a_2^{n} + \alpha\,(a_2^{n} - a_2^{n-1}) \right) + (1-\theta)\, a_2^{n} \tag{11}$$

Once a_1^{n+1} is evaluated from equation (10) similar time marching process applied to Eq. (9b) gives

$$\mathbf{C}_{22}\,(\mathbf{a}_2^{n+1} - a_2^{n}) + \Delta t K_2\,(\Theta_2\,\mathbf{a}_2^{n+1} + (1-\Theta_2) a_2^{n}) =$$

$$\Delta t \bar{\mathbf{f}}_2 - C_{21}(\mathbf{a}_1^{n+1} - \mathbf{a}_1^{n}) + \Delta t K_{21}\,(\Theta_2\, a_1^{n+1} + (1-\theta_2)\, a_1^{n}) \tag{12}$$

The 'splitting' of the vector of the unknown variables $\mathbf{a}$ can be many fold and stability conditions for 'two way' split continue to apply. Thus the variables $\mathbf{a}$ can be subdivided into a series of small partitions and matrices of size $\mathbf{K}_{nn}$ only then require solving at each step of the computation. Indeed the splitting can be continued to the individual components of '$\mathbf{a}$' thus achieving a fully explicit computation.

The order in which the various partitions are dealt with can be varied at alternate steps

Experiments with the procedure are continuing but preliminary tests show that

(a) unconditional stability is readily achieved;

(b) accuracy of computation can vary with the order in which the splitting is carried out.

It appears that a very general and useful computations tool is now available which can be used to reduce substantially the cost of time stepping computation and, if these should be used for iterative purposes, to reduce the cost of such solutions.

The SEE method can in a similar vein be applied to second order equations and its usefulness in that context is now being studied.

If the 'splitting' process is arranged to follow a hierarchical subdivision of variables discussed in the previous section greatly improved accuracy will be achieved for reasons there mentioned.

In Fig. 2, we show the SEE procedure applied as an explicit process in the solution of a one dimensional heat diffusion equation using two alternative sequencing of the partition.

In conclusion, we should mention that the time stepping algorithm developed by Truillo though formally resembling the process described here differ in many substantial respects(in particular, omitting the extrapolation process).[36][37]

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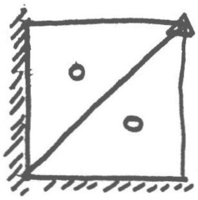
Performance of some 2-D elements for incompressibility

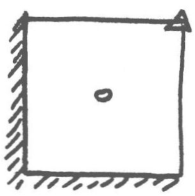
△ 2 D.O.F FOR VELOCITY

○ 1 constraint (integration point or D.O.F for p)



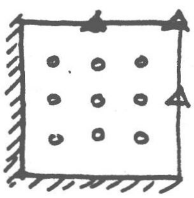
$\beta = \frac{\Delta n}{\Delta m} = \frac{2}{2} = 1$

(COMPLETE LOCKING) LINEAR TRIANGLE



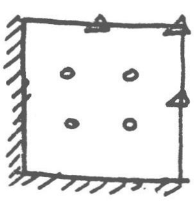
$= \frac{2}{1} = 2$

BILINEAR SELECTIVE INTEGRATION



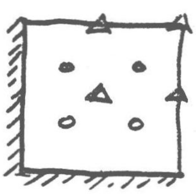
$= \frac{3 \times 2}{9} = 0.67$

FULLY INTEGRATED SERENDIPITY (COMPLETE LOCKING)



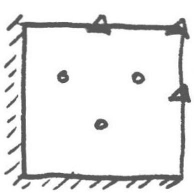
$= \frac{3 \times 2}{4} = 1.5$

REDUCED INTEGRATION SERENDIPITY



$= \frac{4 \times 2}{4} = 2.0$

SELECTIVE INTEGRATION LAGRANGIAN



$= \frac{3 \times 2}{3} = 2.0$

MIXED FORMULATION LINEAR p

FIG 1

FIG 2. Temperature at ℄ of bar (20 linear elements)
x=0 T=1