Ordinary Differential Equations

third edition

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Preface

The third edition of this book differs from its predecessors in a number of ways. The material on Liapunov theory has been expanded moderately. Experience with the book has led me to rearrange some topics—the principal change being that of introducing somewhat earlier the concept of the phase plane; the notion of linear independence appears much earlier in the text. The calculation of series solutions in the neighborhood of a regular singular point has been simplified considerably. The number of exercises has been carefully increased, and some of the proofs have been made more precise.

The text is arranged so that classes unfamiliar with differential equations may begin with Chapter 1. Classes that initially are acquainted with elementary methods of solving differential equations may review the first three chapters quickly before beginning their study of Chapter 4. And those classes for whom the ideas of Chapter 1–3 are quite familiar may well choose to begin their course of study with Chapter 4. Chapters 3 and 6, on physical applications, may be omitted by classes interested only in the mathematical theory, without interrupting the continuity of the text.

This edition, like its predecessors, has the benefit of comments from both teachers and students who have used the text. A useful criticism, for example, came from as far as the National Chiao Tung University where a graduate student, Kun-chou Lin, was troubled by something that in turn troubled the author. My thanks go to all who have contributed the many helpful suggestions.

Special thanks go to Professors Courtney Coleman, David A. Sanchez, and David V. V. Wend who read the manuscript for this edition in its entirety and whose comments were searching, useful, and frequently challenging.

Walter Leighton

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Elementary methods

1 Introduction

Differential equations are equations that involve derivatives. For example, the equations

$$y' = f(x),$$

$$y'' + y = 0,$$

$$y'' = (1 + y'^2)^{1/2}, \tag{1.1}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

are differential equations. The first three of these equations are called ordinary differential equations because they involve the ordinary derivatives of the unknown y. The last equation is an example of a partial differential equation.

We shall be concerned with ordinary differential equations and their solutions.

To solve an algebraic equation, such as

$$x^2 - 3x + 2 = 0, \tag{1.2}$$

we seek a number with the property that when the unknown x is replaced by this number the left-hand member of the equation reduces to zero. In equation (1.2) either the number 1 or the number 2 has this property. We say that this equation has the two solutions 1 and 2. To solve a differential equation we seek to determine not an unknown number but an unknown function. For example, in the equation

$$y'' + y = 0, \tag{1.3}$$

y is regarded as the unknown. To find a solution we attempt to determine *a function defined on an interval* with the property that when y is replaced by this function, the equation reduces to an identity on this interval. It is clear that $\sin x$ is a solution of (1.3) for all values of x, for

$$(\sin x)'' + \sin x \equiv 0 \qquad (-\infty < x < \infty).$$

Similarly, it is easy to verify that $\cos x$ is also a solution of the differential equation (1.3).

Differential equations play a fundamental role in almost every branch of science and of engineering. They are of central importance in mathematical analysis. A differential equation describes the flow of current in a conductor; another describes the flow of heat in a slab. Other differential equations describe the motion of an intercontinental missile; still another describes the behavior of a chemical mixture. Sometimes it is important to find a particular solution of a given differential equation. Often we are more interested in the existence and behavior of solutions of a given differential equation than we are in finding its solutions.

In this chapter we shall begin our study by solving certain simple and important types of differential equations.

The order of a differential equation is the order of the highest derivative that appears in the equation. Accordingly, the first equation in (1.1) is of first order, and the next two equations are of second order. Similarly, the differential equation

$$y''' + y^4 = e^x$$

is of third order, and the equation

$$(y'''')^2 + yy' = 3$$

is of fourth order. The differential equation

$$M(x, y) + N(x, y)y' = 0 \tag{1.4}$$

is of first order. It is frequently useful to rewrite this equation in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{1.4$'$}$$

Thus,

$$(x^2 + y^2)\,dx + 2x\,dy = 0,$$

$$xe^y\,dx + (1 + y)\,dy = 0$$

are differential equations of first order written in the form (1.4)′.

Exercises

1. Verify that if c_1 and c_2 are constants, $c_1 \sin x + c_2 \cos x$ is a solution of the differential equation $y'' + y = 0$.

2. Find by inspection a solution of each of the following differential equations:
(a) $y' - y = 0$;
(b) $y' + 2y = 0$;
(c) $y' = \sin x$.

3. Find by inspection a solution of each of the following differential equations:
(a) $y'' - y = 0$;
(b) $xy' - y = 0$;
(c) $y'' = 0$.

4. Verify that the function $c_1e^x + c_2e^{2x}$ (c_1, c_2 constants) is a solution of the differential equation $y'' - 3y' + 2y = 0$.

5. Verify that $c_1x + c_2x^2$ (c_1, c_2 constants) is a solution of the differential equation $x^2y'' - 2xy' + 2y = 0$.

6. Determine $r(x)$ so that the function $\sin \log x (x > 0)$ is a solution of the differential equation $[r(x)y']' + \dfrac{y}{x} = 0$.

7. Verify that $\sin x$ is a solution of the differential equation $y'^2 + y^2 = 1$.

8. Verify that if c_1 and c_2 are constants and $x > 0$, the function

$$c_1 \sin \frac{1}{x} + c_2 \cos \frac{1}{x}$$

is a solution of the differential equation $(x^2y')' + x^{-2}y = 0$.

9. Verify that $\sin x$, $\cos x$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$, and $\cosh x = \frac{1}{2}(e^x + e^{-x})$ are solutions of the differential equation $y'''' - y = 0$.

10. Show that if a and b are constants, the function $(a^2 \sin^2 x + b^2 \cos^2 x)^{1/2}$ is a solution of the differential equation $y^3(y'' + y) = a^2b^2$ on the interval $-\infty < x < \infty$.

Answers

2. (a) e^x.

6. x.

2 Linear differential equations of first order

A linear differential equation of first order is an equation that can be put in the form

(2.1) $$k(x)y' + m(x)y = s(x).$$

On intervals on which $k(x) \neq 0$, both members of this equation may be divided by $k(x)$, and the resulting equation has the form

(2.2) $$y' + a(x)y = b(x).$$

We shall suppose that $a(x)$ and $b(x)$ are continuous on some interval $a \leq x \leq b$. There are two commonly used elementary methods for solving an equation of the form (2.2).

Method 1. To solve equation (2.2), we may multiply both members of the equation by†

(2.3) $$e^{\int a(x)\,dx},$$

and we have

(2.4) $$[e^{\int a(x)\,dx}y]' = b(x)e^{\int a(x)\,dx}.$$

To solve (2.4) for y, we write

$$e^{\int a(x)\,dx}y = c + \int b(x)e^{\int a(x)\,dx}\,dx \qquad (c \text{ constant}),$$

and, finally,

† By the symbol $\int a(x)\,dx$ is meant any function $A(x)$ such that $A'(x) = a(x)$ $(a \leq x \leq b)$.

$$y = e^{-\int a(x)\,dx}\left[c + \int b(x)e^{\int a(x)\,dx}\,dx\right]. \tag{2.5}$$

The student is advised to use the method described for solving an equation (2.2) rather than formula (2.5).

Example. Solve the differential equation

$$x^2y' + xy = 2 + x^2 \qquad (x > 0). \tag{2.1$'$}$$

We first put this equation in the form (2.2) by dividing it through by x^2:

$$y' + \frac{1}{x}y = \frac{2}{x^2} + 1. \tag{2.2$'$}$$

Here $a(x) = 1/x$ and $b(x) = 1 + 2/x^2$. We note that

$$e^{\int a(x)\,dx} = e^{\int \frac{dx}{x}} = e^{\log x} = x. \tag{2.3$'$}$$

If both members of (2.2)′ are multiplied by x, we have

$$(xy)' = \frac{2}{x} + x. \tag{2.4$'$}$$

From (2.4)′ we have successively

$$xy = 2\log x + \frac{x^2}{2} + c,$$

and

$$y = \frac{2}{x}\log x + \frac{x}{2} + \frac{c}{x} \qquad (x > 0). \tag{2.5$'$}$$

Method 2. This method is of theoretical importance, and it is a method that generalizes to linear differential equations of higher order.

Consider the differential equation

$$y' + a(x)y = b(x) \tag{2.2}$$

and the associated *homogeneous*† equation

$$y' + a(x)y = 0.$$

It is easy to see by substitution that a solution of the latter equation is

$$e^{-\int a(x)\,dx}.$$

† We shall follow the custom of *italicizing* words and phrases that are being defined either explicitly or implicitly in the text.

To complete the solution of (2.2) we introduce a new variable v in (2.2) by means of the substitution

$$y = e^{-\int a(x)\,dx} v, \tag{2.6}$$
$$y' = e^{-\int a(x)\,dx}[v' - a(x)v].$$

Equation (2.2) becomes

$$e^{-\int a(x)\,dx} v' = b(x),$$

and so

$$v' = b(x)e^{\int a(x)\,dx}.$$

The last equation yields

$$v = c + \int b(x)e^{\int a(x)\,dx}\,dx.$$

Using (2.6) we then have

$$y = e^{-\int a(x)\,dx}\left[c + \int b(x)e^{\int a(x)\,dx}\,dx\right],$$

which agrees with equation (2.5).

It will be instructive to apply the second method to the equation

$$x^2y' + xy = 2 + x^2 \qquad (x > 0) \tag{2.1$'$}$$

of the preceding example. The associated homogeneous equation

$$y' + \frac{1}{x}y = 0$$

has the solution

$$e^{-\int a(x)\,dx} = e^{-\int \frac{dx}{x}} = \frac{1}{x}.$$

We accordingly substitute

$$y = \frac{1}{x}v$$

in (2.1)′ obtaining

$$x^2\left(\frac{1}{x}v' - \frac{1}{x^2}v\right) + x\left(\frac{1}{x}v\right) = 2 + x^2 \qquad (x > 0),$$

or

$$v' = \frac{2}{x} + x.$$

Thus,

$$v = 2\log x + \frac{x^2}{2} + c,$$

and

$$y = \frac{2}{x}\log x + \frac{x}{2} + \frac{c}{x} \qquad (x > 0),$$

which is (2.5)′.

The form of the solution of (2.2). We have observed above that

$$e^{-\int a(x)\,dx}$$

is a solution of the homogeneous equation

$$y' + a(x)y = 0. \tag{2.7}$$

Clearly, if c is any constant,

$$ce^{-\int a(x)\,dx} \tag{2.8}$$

is also a solution. We shall learn later that (2.8) is the *general solution* of equation (2.7)—that is, every solution of (2.7) may be put in this form. Suppose now that $y_0(x)$ is some particular solution of the nonhomogeneous equation

$$y' + a(x)y = b(x); \tag{2.9}$$

that is to say,

$$y_0'(x) + a(x)y_0(x) \equiv b(x). \tag{2.10}$$

A substitution into (2.9) reveals that

$$y = y_0(x) + ce^{-\int a(x)\,dx} \tag{2.11}$$

is then also a solution of (2.9) for each value of c. We shall prove later that (2.11) provides the general solution of (2.9).

Example. Consider the differential equation

$$y' + y = 3. \tag{2.12}$$

The corresponding homogeneous equation is

$$y' + y = 0,$$

which has solutions