pre-calculus mathematics

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preface

The primary purpose of this text is to prepare students to study calculus. A closely allied secondary purpose is to teach mathematics. The book has been written with the recommendations of the CUPM in mind; its approach is the study of elementary functions and coordinate geometry. Yet the material here includes *more* than would be required for the intensive Mathematics 0 course recommended by the committee. Most of the material covered in a standard college algebra and trigonometry course is here. (The chief omission is probability.) There is even more trigonometry than is usual in such standard courses. Therefore, courses ranging from the short 45-class-day Math 0 to a year's experience for well prepared high school students are possible using this text.

While topics are covered from a modern point of view, the plan is to build each concept intuitively before making any attempt to be rigorous. In fact, rigor for its own sake has been avoided. The traditional ("new mathematics") treatment of the real number axioms and basic theorems derived from them, found in the beginning of most standard texts, has been placed in the appendix. I have done this so that the student can get to the "meat" of the important mathematical ideas early rather than consuming time on side trips into the foundations of algebra. A brief introduction to complex numbers in terms of ordered pairs of real numbers, does, however, appear in Chapter 1.

On the other hand, people take mathematics courses in mathematics departments from mathematicians, presumably because mathematicians think somewhat differently from normal people. Mathematical thought does have a contribution to make to a person's education. I like to think of myself as a mathematician as well as a teacher. I also believe that some students will not end their mathematical study with this course, or with calculus, but some might even find their way into a linear or an abstract algebra course in the future. This book ought to prepare them, too. Hence, while I have tried not to impose a well-polished logical structure, arrived at after centuries of refinement, on the mathematically unpolished and unrefined student, I see no harm in alluding to this structure from time to time.

In relegating the axiomatic treatment of the real number system to the appendix and in omitting most of the detail in the construction of the complex number system, I frankly assumed that these details were, or will be, covered in some other course for those who are interested.

It is also frankly expected that the student who uses this book will have completed a year and one-half of high school algebra and the usual high school geometry before beginning this course. The first part of Chapter 1 may, therefore, repeat old material. The teacher should decide how much of this chapter is essential. It has been my experience, in over twenty years of teaching mathematics (Does that indicate that I am not to be trusted?), that in both

preface

calculus and in pre-calculus course of this nature and particularly in any further mathematical study, the topics in Chapter 1 and those of Chapter 2 are precisely those which the student needs to learn or review. Most of the terms used in the rest of the book are explained in Chapter 1. Chapter 2 is, admittedly, a bit of a digression. It could be postponed, used as an appendix, assigned as reading, or omitted. I like to do this material in the order in which it appears in the text.

Throughout the book there are a great many worked examples and more than an ample supply of exercises. These exercises may be "drill" for the purpose of fixing concepts, a review of the manipulative aspects of basic algebra, or problems of a theoretical nature which challenge the student to think about the mathematical ideas involved. Answers to many of the exercises appear at the end of the book.

I have used this material both in large classes, taught by the lecture-recitation-section method, and in traditionally organized classes. There is enough material here for a course which meets daily for one semester. A more relaxed pace is possible by omitting some of the material in Chapters 2, 6, or 7, depending on the goals of the particular class, or by meeting three times a week for two semesters or two quarters. Shorter courses can also be constructed using this material. Some possible uses for one-quarter or one-semester courses in *pre-calculus mathematics* or in *elementary functions and coordinate geometry* are suggested in the table on page vii, the selection of optional sections or chapters to be determined by the needs of the individual class.

I wish to express my deep appreciation to the following people who reviewed the manuscript and made many suggestions for improvement. Most of their suggestions were gratefully accepted; any remaining defects are clearly not their responsibility. Professor Gerald M. Armstrong of Brigham Young University, Provo, Utah; Professor Robert W. Negus, Rio Hondo Junior College, Whittier, Calif.; Professor Constantine Georgakis, DePaul University, Chicago, Ill.; Professor Charles H. Neumann, Alpena Community College, Alpena, Mich.; Professor John Morton, Baylor University, Waco, Tex.; Professor Don E. Edmondson, University of Texas, Austin, Tex.; and Professor Harold S. Engelsohn, Kingsborough Community College, Brooklyn, N.Y. My thanks also go to Mary Heywood who typed most of the manuscript, and to Frederick C. Corey, Gary Ostedt and the staff of John Wiley & Sons, for their considerable help and valuable suggestions.

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Provo, Utah H. G. M.

POSSIBLE COURSE OUTLINES

pre-calculus mathematics		elementary functions and coordinate geometry	
Daily for 1 Quarter	Daily for 1 Semester	3 Days/Week For 1 Quarter	3 Days/Week For 1 Semester
CHAPTER 1: Section 1.5 is optional.	CHAPTER 1:	CHAPTER 1: Concentrate on Sections 1.6–1.9.	CHAPTER 1: Cover Sections 1.1–1.5 lightly. Concentrate on Sections 1.6–1.9.
CHAPTER 2: is optional	CHAPTER 2: is optional.	CHAPTER 2: omit.	CHAPTER 2: omit.
CHAPTER 3:	CHAPTER 3:	CHAPTER 3:	CHAPTER 3:
CHAPTER 4:	CHAPTER 4:	CHAPTER 4: Combine Sections 4.4 & 4.5.	CHAPTER 4:
CHAPTER 5: Omit Sections 5.1 & 5.12. Combine Sections 5.2 & 5.3 with Section 5.11.	CHAPTER 5: Sections 5.1 & 5.12 are optional.	CHAPTER 5: Omit Sections 5.1 & 5.12. Combine Sections 5.2 & 5.3 with Section 5.11.	CHAPTER 5: Sections 5.1 & 5.12 are optional.
CHAPTER 6: is optional.	CHAPTER 6:	CHAPTER 6: omit.	CHAPTER 6: is optional.
CHAPTER 7: is optional.	CHAPTER 7: Sections 7.5, 7.7, & 7.8 are optional.	CHAPTER 7: Sections 7.5, 7.7, & 7.8 are optional. Combine Sections 7.1 & 7.6 and Sections 7.3 & 7.4.	CHAPTER 7: Sections 7.5, 7.7, & 7.8 are optional.

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fundamentals

1

In this chapter we shall consider some fundamental ideas which form the basis for the vocabulary and development of the rest of the book and for further study of mathematics. While you are not expected to completely master all of the details of the concepts presented here, you should be familiar enough with them to be able to apply them to your subsequent mathematical work.

1.1. sets: a brief look

The notation and terminology of set theory forms the basic vocabulary of all present-day mathematics. Developed in the nineteenth century principally by George Boole (1815–1864) and Georg F. L. P. Cantor (1845–1918), it has had profound effect upon both the method and content of twentieth century mathematics. The inclusion of certain aspects of this topic in elementary and secondary school mathematics courses and texts seems, at times, the sole criterion for judging the class to be "the new math."

It is fortunately not necessary for our purposes to travel the long and perilous road of a thorough treatment of set theory. We adopt an unsophisticated approach which will enable us to make use of its helpful language in the rest of this book.

A **set** in mathematics means about what it does in ordinary usage in English; i.e., a collection of things under some unifying property: a set of china, a chess set, a set of golf clubs, a set of carving knives, "the jet set," etc. Besides *collection*, other synonyms for set are *herd* (of cattle), *gaggle* (of geese), *flock* (of sheep), *clique*, and so on. The individual objects which make up the set are called its **elements** or **members**.

fundamentals

In this book, sets generally will be denoted by capital letters, $A, B, C, \ldots$ etc., while the elements are designated with lower case letters, $a, b, x, y, z, \ldots$. The sentence

The object x belongs to (is an element or member of) the set S,

will be abbreviated as

$$x \in S$$

The negation of this will be written $x \notin S$; that is, "x does not belong to S."

We have not given a mathematical definition of a set; indeed, we take *set* to be a "*primitive term*." You will gain an intuitive conception of sets by looking at the following examples.

examples of sets

(1) The set of all students at this school.
(2) The set of all planets in our solar system.
(3) The set of all red-headed coeds taking this course.
(4) The set of all letters in the English alphabet.
(5) The set of all citizens of the United States.
(6) The set of all months in the year.
(7) The set N of all natural numbers.
(8) The set Z of all integers: positive, negative, or zero.
(9) The set Q of all rational numbers.
(10) The set R of all real numbers.
(11) The set C of all complex numbers.
(12) The set of all real numbers x satisfying $0 \leq x \leq 1$. This set is called the **unit interval.**
(13) The set of all former presidents of the United States.
(14) The set of all possible bridge hands.
(15) The set of all the letters in the word "Mississippi." ●

We shall discuss the sets listed in (7)–(11) above in greater detail in the next section.

In describing certain sets, some additional notation is frequently handy. One generally uses braces around either a list of the elements of the set, the *roster notation*, as

$$A = \{a, b, c\}$$

for the set consisting of the first three letters of the (English) alphabet; or, if all the elements x of the set have some distinguishing property $P(x)$ which they satisfy, we use the *set builder notation*

$$\{x | x \text{ has } P(x)\}$$

For example, the set of all the former presidents of the United States could be written as

{Washington, Adams, Jefferson, . . . , Eisenhower, Kennedy, Johnson, . . .}

or

$$\{p | p \text{ is a former president of the United States}\}$$

sets: a brief look

Two sets A and B are called equal,

$$A = B$$

if and only if they contain precisely the same elements. Thus, $\{2,4,6,8\} = \{8,8,6,2,4,\}$, even though the symbol 8 appears twice in the second set. The set (15) of all letters in the word "Mississippi" is most economically written $\{M,i,s,p\}$.

A set A is called a **subset** *of a set B,*

$$A \subseteq B$$

provided that each member of A also belongs to B; i.e., $x \in A$ implies $x \in B$. Note that $A = B$ if, and only if, $A \subseteq B$ and $B \subseteq A$. Further note that for every set S, $S \subseteq S$. *If $A \subseteq B$, but $A \neq B$, we often write $A \subset B$ and call A a* **proper subset** *of B.*

It is also convenient to consider a special set, **the empty set** $\varnothing$, which has no elements; it is empty. From the definition of subset, we see that $\varnothing \subseteq S$ for every set S.

One should be very careful to avoid one of the perils of set theory: to distinguish between an object x and the set $\{x\}$ whose only element is x. This is somewhat akin to differentiating between a student and a special reading course for which he is the only registrant, or between a can containing one peanut and the peanut. All the sets considered in this text will be subsets of some large universal set—usually the set of real numbers—called the **universe,** U.

Use of set theoretic ideas is frequently aided by means of pictorial representations called *Venn diagrams*. For example, if A and B are sets (subsets of U) such that $A \subset B$, we can picture this in the Venn diagram of Figure 1.1.1.

That part of the universe U which is not in the set B is called the **complement** of B and is denoted by $\tilde{B}$. In Figure 1.1.1, $\tilde{B}$ is everything outside of the large circle. The annular region inside the circle B (in Fig. 1.1.1) but outside of the circle A is the complement of A in B and is denoted by $B\backslash A$ (some authors use $B - A$).

Given two sets S and T we can consider new sets formed by them. We give the formal definitions first and then sketch Venn diagrams which illustrate them.

(1) The **complement** of T in S, $S\backslash T$ is defined by

$$S\backslash T = \{x|x \in S \text{ and } x \notin T\}$$

(2) The **union** of S and T, $S \cup T$ is defined by

$$S \cup T = \{x|x \in S \text{ or } x \in T\}$$

Here "or" means "either $x \in S$ or $x \in T$ or x is in *both*."

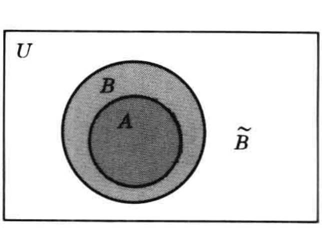

figure 1.1.1

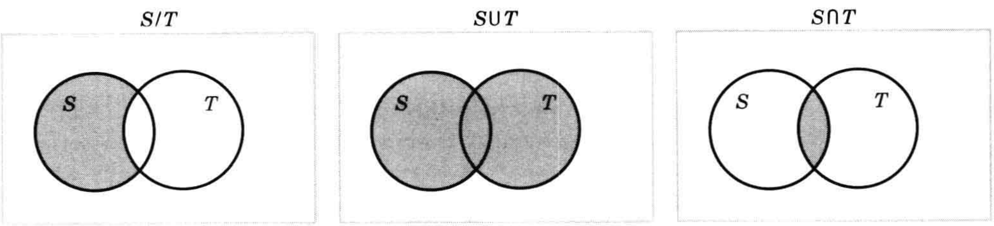

figure 1.1.2

(3) The **intersection** of S and T, $S \cap T$ is defined by

$$S \cap T = \{x | x \in S \text{ and } x \in T\}$$

examples

(16) Let $S = \{a,b,c,d\}$, $T = \{b,d,f,h\}$, $R = \{a,e,i\}$. Then we have the following.

Union:

$S \cup T = \{a,b,c,d,f,h\}$
$S \cup R = \{a,b,c,d,e,i\}$
$T \cup R = \{a,b,d,e,f,h,i\}$

Intersection:

$S \cap T = \{b,d\}$
$S \cap R = \{a\}$
$T \cap R = \varnothing$

Complement:

$T \backslash S = \{f,h\}$
$S \backslash T = \{a,c\}$
$S \backslash R = \{b,c,d\}$
$R \backslash S = \{e,i\}$

(17) Let

$P = \{p | p$ is a former president of the United States$\}$
$A = \{l | l$ is a world politician (leader) who was assassinated$\}$
$V = \{v | v$ is a former vice president of the United States$\}$

Then we have

$P \cap V = \{$Adams, Jefferson, Van Buren, Tyler, Fillmore, Andrew Johnson, Arthur, T. Roosevelt, Coolidge, Truman, Lyndon Johnson$\}$
$P \cap A = \{$Lincoln, Garfield, McKinley, Kennedy$\}$

(18) Let T be the set of positive integral multiple of 2,

$$T = \{2\text{n} | n \in N\}^*$$

*Remember that N denotes the set of Example 7 consisting of the natural numbers

$$N = \{1, 2, 3, 4, \ldots\}$$

and S be the set of positive integral multiples of 3, $S = \{3k | k \in N\}$. We have $S \cup T = \{x | x = 2n \text{ or } x = 3n, \text{ some } n \in N\}$; i.e.,

$$\{2,3,6,8,9,10,12, \ldots\}$$

$S \cap T = \{x | x = 6k, k \in N\}$; i.e.

$$\{6,12,18, \ldots\} \quad \bullet$$

Consider an arbitrary set X. The set of all subsets of X is denoted by 2^X and is called the **power set** of X. For example, let $X = \{\text{Tom, Dick, Harry}\}$. The the subsets of X are X, $\varnothing$, $\{\text{Tom}\} = T$, $\{\text{Dick}\} = D$, $\{\text{Harry}\} = H$, $\{\text{Tom, Dick}\} = R$, $\{\text{Tom, Harry}\} = S$, and $\{\text{Dick, Harry}\} = W$. Thus $2^X = \{X,\varnothing,T,D,H,R,S,W\}$.

If $S \cap T = \varnothing$, then S and T are called **disjoint**.

Another method by which new sets are obtained from old ones involves the **Cartesian* product** of two sets. If A and B are two sets, then we define

$$A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$$

This set will be exploited in the particular case where A and B are each the set R of real numbers.

example

(19) Let $A = \{a,p,m,e\}$, $B = \{p,e,a,r\}$.

Then
$$\begin{aligned} A \cup B &= \{a,p,m,e,r\} \\ A \cap B &= \{a,p,e\} \\ B \setminus A &= \{r\} \\ A \setminus B &= \{m\} \\ A \times B &= \{(a,p), (a,e), (a,a), (a,r), (p,p), (p,e), (p,a), (p,r), \\ &\quad (m,p), (m,e), (m,a), (m,r), (e,p), (e,e), (e,a), (e,r)\} \end{aligned}$$

Note that none of (m,m), (r,r), (p,m) belongs to $A \times B$. ●

These operations on sets have many formal consequences, some of which are included in the exercises. We forego, however, a complete study of the "algebra of sets."

exercises

1. Use the roster method to describe each of the following sets.

(a) the set of all vowels in the English alphabet
(b) $\{x | x \text{ is an integer between } -3 \text{ and } 3\}$
(c) $\{x | x^2 - 3x + 2 = 0, x \text{ a real number}\}$
(d) $\{x | x + 1 = 2x, x \text{ an integer}\}$
(e) $\{z | z^4 = 16, z \text{ a real number}\}$

2. Given the following four sets,

$$A = \{1, 2\} \; B = \{1, 2, 3\} \; C = \{2, 3\} \; D = \{3\}$$

*After the French mathematician René Descartes (1596–1650).

describe:

(a) $A \cup B$
(b) $A \cap B$
(c) $B \cap C$
(d) $B \setminus A$
(e) $B \setminus D$
(f) $A \times B$
(g) $A \times C$
(h) $A \times (B \times C)$
(i) $A \cup (B \cup C)$
(j) $A \cap (B \cup C)$
(k) $A \cup (B \cap C)$
(l) $A \cap (B \cap C)$
(m) 2^A
(n) 2^B
(o) 2^C
(p) 2^D

3 Determine each of the following subsets of the set P of all former presidents of the United States (see any almanac).

(a) $\{p | p \text{ is still living}\}$
(b) $\{p | p \text{ served more than two terms}\}$
(c) $\{p | p \text{ served more than one term}\}$
(d) $\{p | p \text{ served two nonconsecutive terms}\}$
(e) $\{p | p \text{ is a woman}\}$

4. Given the sets of Exercise 2, which of the following are correct?

(a) $A \subseteq B$
(b) $B \subseteq C$
(c) $D \subseteq C$
(d) $A \cup C = B$
(e) $A \cap D = \varnothing$
(f) $A \subseteq A \times B$
(g) $A \times D \subseteq A \times B$
(h) $A \cup B = B \cup A$

5. Use Venn diagrams and shade the region $A \cap B$ when each of the following is true.

(a) $A \subset B$
(b) $B \subset A$
(c) $A \cap B = \varnothing$
(d) $A = B$
(e) $A = \tilde{B}$
(f) $A \setminus B = \varnothing$

6. Use Venn diagrams and shade the region $A \cup B$ for each of the situations in Exercise 5.

7. (a) Write down all the subsets of the set $\{x,y,z\}$. (There are eight of them.)
(b) How many elements are there in 2^X if X has n elements, n a positive integer?

8. Using Venn diagrams verify the following concerning any subsets A, B, C of the universe U.

(a) $A \cup B = B \cup A$ } *commutative laws*
(b) $A \cap B = B \cap A$ }
(c) $A \cup (B \cup C) = (A \cup B) \cup C$ } *associative laws*
(d) $A \cap (B \cap C) = (A \cap B) \cap C$ }
(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ } *distributive laws*
(f) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ }
(g) $\varnothing \subseteq A \cap B \subseteq A \subseteq A \cup B$
(h) $\varnothing \subseteq A \cap B \subseteq B \subseteq A \cup B$

9. Use the formal definitions given to prove each of the following statements concerning sets A, B, C.

(a) $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$.
(b) $A \subseteq \varnothing$ if, and only if, $A = \varnothing$.
(c) $B \setminus A = \tilde{A} \cap B$.
(d) $A \subseteq B$ if, and only if, $\tilde{B} \subseteq \tilde{A}$.

10. Use Venn diagrams to indicate that the following *De Morgan laws* hold for arbitrary subsets A, B, and X of U.

(a) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$
(b) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

11. Answer each of the following questions.

(a) If A is any set, is $A \subseteq A$?
(b) If A is any set, is $A \in A$?
(c) If A is any set, does $A \subseteq \varnothing$?
(d) How many elements are there in $\varnothing$?
(e) How many elements are there in $\{\varnothing\}$?
(f) If A is any set, is $\varnothing \in A$?
(g) If A is any set, is $\varnothing \subseteq A$?
(h) For any set A, $A \cap \tilde{A} =$?
(i) What is $\tilde{\varnothing}$?
(j) For any set A, what is $(\tilde{\tilde{A}})$?
(k) For any set A, $A \cup A =$?
(l) For any set A, $A \cap A =$?

12. Prove each of the following formally.

(a) $(\widetilde{A \cap B}) \subseteq \tilde{A} \cup \tilde{B}$.
(b) If $A \subseteq B$, then $A \cup B = B$.
(c) If $A \cap B = \varnothing$, then $A \cup \tilde{B} = \tilde{B}$.
(d) If $A \subseteq B$, then $A \cap B = A$.

1.2. sets of numbers

In this text we shall, for the most part, be concerned with sets of numbers rather than sets in general. In particular we shall frequently use the set of real numbers for our universe. As was the case with our study of sets, we shall approach the real numbers on a largely intuitive basis. A rigorous development of the real number system more properly belongs to advanced courses. One can refer to such texts as E. Landau, *Foundations of Analysis*, Chelsea, 1951, or J. M. H. Olmsted, *The Real Number System*, Appleton-Century-Crofts, 1962, for such a development. In Appendix I we list the field axioms and basic theorems which one will meet in the usual Intermediate Algebra course development of the real number system.

We take the concept of a **real number** to be a *primitive term*—we don't try to define it. We describe some of the properties of real numbers. To begin with, consider the important subsets of the set R of real numbers.

Natural Numbers. The set N of natural numbers consists of those numbers which arise naturally from counting, namely, 1,2,3,4, ...

$$N = \{1,2,3, \ldots\}$$

Integers. The set Z of integers consists of all the natural numbers, their negatives and the number zero. Thus,

$$Z = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

The *natural numbers* are often referred to as *positive integers*, and the natural numbers with zero included as *non-negative integers*.

Rational Numbers. The set Q of rational numbers consists of those real numbers which can be expressed as the quotient of the two integers. Hence,

$$Q = \{x | x = p/q, p \in Z, q \in Z, q \neq 0\}$$