Erich Hecke

Lectures on Dirichlet Series, Modular Functions and Quadratic Forms

Vandenhoeck & Ruprecht

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Lectures on Dirichlet Series, Modular Functions and Quadratic Forms

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in collaboration with Wilhelm Maak

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Foreword

Erich Hecke was granted sabbatical leave from the University of Hamburg in the first half of 1938. This leave was to be spent at the Universities of Michigan and Princeton. While in America, Hecke held lectures based on his investigations conducted between 1935 and 1937. These had been published in Mathematische Annalen Bd 112 (1936) and Bd 114 (1937). For the lectures Hecke had prepared notes in English, and these survived. In addition notes taken by one of the audience were subsequently circulated in planographed form. Hecke's manuscript and the notes are substantially in agreement. The differences are almost entirely limited to form and language. The version of the lectures published here is based on both the manuscript and the lecture notes. Here the reader will find no new results over and above in the papers to which reference has already been made. The major difference from the publications is that in the present book considerable emphasis has been placed on elucidating Heckes basic ideas.

The manuscript has been prepared for press by Dr. R.-D. Kulle, who is also responsible for several improvements.

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Introduction

1. The Riemann zeta function

The starting point of analytic theory of numbers can be traced back to the paper of *Riemann 1859*. He states the following point of view: An arithmetical problem of the kind: "To determine an arithmetically defined function $a(n)$ of an integer n in its dependence on n" is equivalent with a problem: "To investigate an analytical function $\sum_{n=1}^{\infty} a(n)f_n(z)$ with suitable $f_n(z)$", specially by its behaviour on its singular points.

The problem of the distribution of the prime numbers p thus can be associated with the function $\sum_p z^p$ for example; but we are not able to examine this function or to state a connex between this function and other already known functions. Instead of that Riemann considers the function $\sum_p p^{-s}$ defined for all s with $Re(s) > 1$. He shows by using a statement of *Euler* that the function can be reduced into a simpler function, namely

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad Re(s) > 1\,.$$

For there exists the so called Euler product

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}\,,$$

and from this follows

$$\log\zeta(s) = -\sum_p \log(1 - p^{-s}) = \sum_p p^{-s} + \tfrac{1}{2}\sum_p p^{-2s} + \cdots = \sum_p \frac{1}{p^s} + g(s)\,,$$

where $g(s)$ is determined by Dirichlet series which is obviously regular in the half-plane $\mathrm{Re}(s) > \frac{1}{2}$. Therefore the singular points of $\sum_p p^{-s}$ and $\log\zeta(s)$ in the $\mathrm{Re}(s) > \frac{1}{2}$ are the same and the investigation of $\sum_p p^{-s}$ in this domaine can be replaced by the investigation of $\log\zeta(s)$. But the singular points of $\log\zeta(s)$ are the singular points of $\zeta(s)$ and the zeros of $\zeta(s)$. Now the point $s = 1$ must be singular, because it can easily be proved, that $\zeta(s) \to \infty$ if $s \to 1$.

Then $\sum_p p^{-s}$ must have the singular point $s = 1$, and thus the number of terms in this sum must be infinite, that means, *the number of primes is infinite.*

But furthermore it can be also proved that $(s-1)\zeta(s)$ is regular in the half-plane $\mathrm{Re}(s) > 0$, and thus the only singular points of $\sum_p p^{-s}$ in $\mathrm{Re}(s) > \frac{1}{2}$ are

1) $s = 1$, where $\sum_p p^{-s} - \log \frac{1}{s-1}$ is regular
2) all zeros of $\zeta(s)$ in the domain $\mathrm{Re}(s) > \frac{1}{2}$.

Now the famous *hypothesis of Riemann* arises:

$$\zeta(s) \neq 0 \text{ in } \mathrm{Re}(s) > \tfrac{1}{2}.$$

And then Riemann tries to study the behaviour of $\zeta(s)$ in order to obtain some new theorems concerning the distribution of prime numbers. And indead; if $\pi(x)$ denotes the number of primes $\leq x$, than $\pi(x)$ can be expressed in the form

$$\pi(x) = \int^{x} \frac{dn}{\log n} + e(x)$$

where $e(x)$ depends on the zeros of $\zeta(s)$ and is of the order $O\left(x^{\frac{1}{2}+\varepsilon}\right)$, if the Riemann hypothesis is true. Riemann gives only a scetch of the prove, many ideas yet were wanting, for example in order to prove correctly this theorem it was necessary to find a theory of entire functions of finite genus, stated later by Hadamard and de la Vallée-Poussin. The first important step has been made already by Riemann himself, proving that $\zeta(s)$ can be continuated analytically in the total finite plane and that $(s-1)\zeta(s)$ is an entire function of s.

Together with that Riemann proves also the important functional equation of $\zeta(s)$:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ is invariant under $s \to 1-s$.

The present investigation springs from the connection between the Riemann $\zeta(s)$, the theta function $\vartheta(\tau)$ and the functional equation for $\zeta(s)$.

The first proof of the functional equation is based upon a general statement, transforming a Dirichlet series into a power series by means of the Γ-integral. According the representation of $\Gamma(s)$

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx, \quad \mathrm{Re}(s) > 0$$

it results

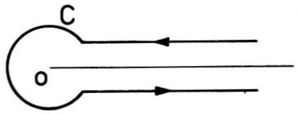
1) $\quad l^{-s}\Gamma(s) = \int_0^\infty e^{-lx} x^{s-1} dx \qquad$ for every $l > 0$.

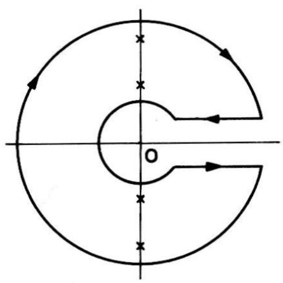
Hence, for $0 < l_1 < l_2 < \dots$

$$\left(\sum_{n=1}^{\infty} a_n l_n^{-s}\right)\Gamma(s) = \int_0^{\infty} x^{s-1}\left(\sum_{n=1}^{\infty} a_n e^{-l_n x}\right)dx,$$

of course under certain asymptions about the convergence of the Dirichlet series.
In particular, for $l_n = n$, $a_n = 1$

$$\zeta(s)\Gamma(s) = \int_0^{\infty} x^{s-1}\left(\sum_{n=1}^{\infty} e^{-nx}\right)dx = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\,dx.$$





The integral is then transformed by the methods of the classical theory of analytical functions, into one in the complex x-plane. The contour is to include the singular points, 0 and ∞, of the integrand as well as a cut connecting these points. But it does not include other zeros of $e^x - 1$.

$$\zeta(s)\Gamma(s) = \frac{e^{-i\pi s}}{2i\sin\pi s}\int_C \frac{x^{s-1}}{e^x - 1}\,dx.$$

The new integral is defined for all s and is an analytic, integral function of s. The functional equation follows by applying Cauchy's theorem for $\mathrm{Re}(s) < 0$. In fact, one obtains

$$\zeta(s)\Gamma(s) = \frac{2^{s-1}\pi^s}{\cos\frac{\pi s}{2}}\sum_{k=1}^{\infty} k^{s-1} = \frac{2^{s-1}\pi^s}{\cos\frac{\pi s}{2}}\zeta(1-s),\ \mathrm{Re}(s) < 0.$$

This may be expressed by the assertion $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is invariant under $s \to 1 - s$.

This suggested a new proof to Riemann, by substituting $l \to n^2 \pi, s \to \frac{s}{2}$ in 1)

$$\pi^{-\frac{s}{2}} n^{-s} \Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-n^2 x \pi} x^{\frac{s}{2}-1} dx$$

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \left(\sum_{n=1}^\infty e^{-\pi n^2 x}\right) dx .$$

And now the functional equation of $\zeta(s)$ is a simple consequence of the transformation formula for the theta-function

$$\vartheta(\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 \tau} = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi n^2 \tau}, \quad Im(\tau) > 0 ,$$

$$\vartheta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\vartheta(\tau) .$$

For

$$\int_0^\infty x^{\frac{s}{2}-1} \left(\sum_{n=1}^\infty e^{-\pi n^2 x}\right) dx = \int_1^\infty x^{-\frac{s}{2}-1} \left(\sum_{n=1}^\infty e^{\frac{-\pi n^2}{x}}\right) dx + \int_1^\infty x^{\frac{s}{2}-1} \left(\sum_{n=1}^\infty e^{-\pi n^2 x}\right) dx.$$

But

$$\sum_{n=1}^\infty e^{-n^2 \pi x} = \tfrac{1}{2}\left(\vartheta(ix) - 1\right).$$

Applying the functional equation for $\vartheta(ix)$ gives the functional equation for $\zeta(s)$.

The functional equation for $\zeta(s)$ is very important, particularly because $\zeta(s)$ is uniquely determined by it under certain assumptions of regularity. The proof by Mr. Hamburger in 1921 is simple. One expresses the partial sum of the coefficients of a Dirichlet series by means of an integral to which the functional equation is applied. The proposition then results.

But the reasoning is not satisfactory. We do not comprehend the genuine reason for the existence of the functional equation since Riemann's method of replacing s by $\frac{s}{2}$ is no more than a verification, a device to produce an isolated fact. Nor do we see why the assumptions of the functional equation and of regularity are so strong as to imply the uniqueness of the solution. Indeed, the theorem can be greatly extended and then it can be interpreted more intelligibly.

Furthermore, there are many other Dirichlet series $\varphi(s)$ satisfying a similar type of functional equation:

$$\left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)^a \Gamma\left(\frac{s}{2}\right)^b \Gamma\left(\frac{s+1}{2}\right)^c \varphi(s)$$

is invariant under $s \to k - s$ for given $\lambda > 0, k > 0$.

For example:

1) $a = 1, b = c = 0, \lambda = |\sqrt{D}|$ for the zeta-function of an imaginary quadratic field $K(\sqrt{D})$.
2) $a = 0, b = 2, c = 0, \lambda = 2|\sqrt{D}|$ for the zeta-function of a real quadratic field $K(\sqrt{D})$.
3) $a = b = 0, c = 1$ or $a = c = 0, b = 1$ for the L-series $L(s, \chi)$ mod m in the rational field with primitiv character. Here is $\lambda = 2\sqrt{m\pi}$ and $\chi(-1) = -1$ for the first case, $\chi(-1) = 1$ for the second case.

In 1), 2), 3) is $k = 1$.

All these series have Eulerian products of the type

$$\varphi(s) = \prod_p (1 - a_p p^{-s})^{-1}(1 - \tilde{a}_p p^{-s})^{-1}$$

where p ranges through the prime numbers, a_p, $\tilde{a}_p$ being coefficients independent of s. The functional equation is always proved by reduction to multiple theta-functions or similar series.

By a *simple Γ-factor* we shall mean one with one of $a, b, c = 1$, the others $= 0$, or $a = 0$ and $b = c = 1$.

Let us recall the relation

$$2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \pi^{\frac{1}{2}}\Gamma(s).$$

There is a fundamental clarification of the relations existing between the Dirichlet series which satisfy a functional equation with a simple Γ-factor and the general automorphic functions, in particular, modular functions.

At first I will report the ideas and the character of the theory by examples without complet proofs.

It will be possible to determine uniquely many of the Dirichlet series of arithmetic by means of functional properties, for example $\zeta(s)$ and the zeta-functions of the imaginary quadratic fields.

Further, it will be proved that the Euler product is the counterpart of a new algebraic property of the modular functions; indeed many of these Dirichlet series have an Eulerian product – a fact not hitherto known.

Finally, by application of these theorems to multiple theta-series, there results a set of new and remarkable assertions concerning quadratic forms with integral coefficients. These are of a purely arithmetical character, unknown up to now.

It is possible to give the complet theory with proofs in few lectures, but I will try to give the main-ideas and the character of the results which are partly very unexpected.

2. Associated Dirichlet series and power series

Now for the connection between these ideas: We consider the expression

$$(1) \quad R(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\varphi(s)$$

formed with an arbitrary function $\varphi(s)$ and we will say

"$\varphi(s)$ has signature (λ, k, γ)"

if with given $\lambda > 0$, $k > 0$, γ an arbitrary constant

1) $R(s) = \gamma R(k-s)$.
2) $(s-k)\varphi(s)$ is an integral function of finite genus.
   (To say that $f(s)$ has finite genus means that there is a constant M such that $|f(s)| \leqq e^{|s|^M}$ as $|s| \to \infty$.)
3) $\varphi(s)$ is a Dirichlet series which converges somewhere.

Obviously $\gamma = \pm 1$. First we assume $\varphi(s)$ to be a Dirichlet series of the form

$$\sum_{n=1}^{\infty} a_n l_n^{-s}, \; 0 < l_1 < l_2 < \ldots \to \infty$$

and we introduce the assumption that the l_n's are integers.

We associate to the Dirichlet series

$$\varphi(s) = \sum_{n=1}^{\infty} a_n l_n^{-s},$$

the series

$$(2) \quad f(\tau) = a_0 + \sum_{n=1}^{\infty} a_n e^{\frac{2\pi i l_n \tau}{\lambda}}$$

where the coefficient a_0 is determined afterwards by means of the residue of $\varphi(s)$ at $s = k$. The function $f(\tau)$ is regular in the upper half-plane of the complex variable τ. This *formal correspondence* can be realized by an analytic functional transformation

$$R(s) = \int_0^{\infty} \Big(\sum_{n=1}^{\infty} a_n e^{-\frac{2\pi l_n \tau}{\lambda}}\Big) x^{s-1}\, dx = \int_0^{\infty} (f(ix) - a_0) x^{s-1}\, dx.$$

But also conversly by Mellin's formula

$$(3) \quad f(ix) - a_0 = \frac{1}{2\pi i} \int_{Re(s) = \sigma_0} R(s) x^{-s}\, ds.$$

From both formulas it follows that

1) If

$$R(s)=\gamma R(k-s), \text{ then } \frac{f\left(-\frac{1}{\tau}\right)}{(-i\tau)^k}=\text{-}\gamma f(\tau)$$

where $a_0=\gamma\left(\frac{2\pi}{\lambda}\right)^{-k}\Gamma(k)\cdot\text{Residue } \varphi(s) \text{ at } s=k.$

2) If

$$f\left(-\frac{1}{\tau}\right)=(-i\tau)^k\cdot\gamma f(\tau), \text{ then } R(s)=\gamma R(k-s).$$

This is valid if $f(\tau)$ does not increase too rapidly as $\tau\to 0$ so that $\int\limits_0^\infty$ has a significance.

The demand that $\varphi(s)$ be a special Dirichlet series $\sum\limits_{n=1}^{\infty} a_n n^{-s}$ is precisely equivalent to the assertion that $f(\tau)$ defined by (3) is periodic with period λ.

The **first Main Theorem** states: *To every Dirichlet series with signature* (λ, k, γ) *there corresponds one and only one function* $f(\tau)$ *under a linear functional transformation with the properties*

a) $f(\tau+\lambda)=f(\tau)$ *and* $f(\tau)$ *is regular in* $e^{\frac{2\pi i\tau}{\lambda}}$ *in the upper half-plane of* τ.

b) $\frac{f\left(-\frac{1}{\tau}\right)}{(-i\tau)^k}=\gamma f(\tau)$

c) $f(x+iy)=O(y^{\text{const}})$ *as* $y\to+0$ *uniformly in real* x.

Inversely, to every function $f(\tau)$ *satisfying a), b), c) there corresponds a Dirichlet series of signature* (λ, k, γ).

The quotient of two such functions $f(\tau)$ with the same (λ, k, γ) is obviously an analytic function of τ invariant under the *group* $\mathfrak{G}(\lambda)$ of substitutions generated by

$$U\colon \tau\to\tau+\lambda, \qquad T\colon \tau\to-\frac{1}{\tau}.$$

Thus every linear functional theorem concerning such automorphic functions can be translated into a theorem on the Dirichlet series for (λ, k, γ).

For example – and this is the first kind of result –

1) $0<\lambda<2$. $\mathfrak{G}(1)$ is the modular group. Therefore, a Dirichlet series with signature $(1, k, \gamma)$ exists if and only if k is even integer, $k\geqq 4$, $\gamma=(-1)^{\frac{k}{2}}$.

In this case the number of linearly independent solutions of the problem with given $(1, k, \gamma)$ is finite and equal to

$$\left[\frac{k}{12}\right] + 1, \text{ if } k \not\equiv 2 \pmod{12}$$

$$\left[\frac{k}{12}\right] \quad , \text{ if } k \equiv 2 \pmod{12}.$$

($[a]$ is the greatest integer less than or equal to a.)
Obviously there is always a solution

$$\varphi(s) = \zeta(s) \cdot \zeta(s - k + 1)$$

and there are no other solutions for $k < 12$.

A Dirichlet series for $(1, k, \gamma)$ has an infinity of zeros on the mean line $Re(s) = \frac{k}{2}$ if and only if it is regular at $s = k$.

If $0 < \lambda < 2$ there exists only a finite number of linearly independent functions $\varphi(s)$ for (λ, k, γ). But this number is greater than zero only for a *discrete* set of values of λ and k. Thus if $\lambda = 1$, k must be an even integer.

2) $\lambda = 2$. $\mathfrak{G}(2)$ is a subgroup of the modular group of index 3 associated with the theta-functions. For the signature $(2, k, 1)$ there exists a solution for all positive k, the number of linearly independent ones being finite and equal to

$$\left[\frac{k}{4}\right] + 1.$$

Hence for $k < 4$, there is a unique solution. For example

$$k = \tfrac{1}{2} : \varphi(s) = \zeta(2s).$$

In this case $\varphi(\frac{1}{2} s)$ is also a special Dirichlet series. That means that in $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, the a_n's are zero if n is not a square. This seems to be the suitable form of the theorem concerning $\zeta(s)$.

$$k = 1: \ \varphi(s) = \text{zeta-function of the field } K(\sqrt{-1}) \text{ equals } \zeta(s) \cdot L(s).$$
$$k = 2: \ \varphi(s) = \zeta(s)\zeta(s-1)(1 - 2^{2-2s}).$$

In all cases $k < 4$, the Dirichlet series with signature $(2, k, 1)$ are associated with $\vartheta^{2k}(\tau)$. And the function $\zeta(s)$ is an element of a set of series depending continuously on a parameter k. Each of these series has an infinity of zeros on its mean line $Re(s) = \frac{k}{2}$.

For the signatur $(2, k, -1)$ the number of linearly independent solutions is equal to

$$\left[\frac{k-2}{4}\right]+1.$$

3) If $\lambda > 2$, there exists for every positive k an infinite number of linearly independent Dirichlet series with the same signature (λ, k, γ). It follows that there is an infinite number of Dirichlet series satisfying the functional equation of the zeta-function of the imaginary quadratic fields $K(\sqrt{-D})$ of discriminant $-D$, $D > 4$.

3. The Euler product

To determine such a Dirichlet series uniquely, one must consider additional conditions.

1) The postulate of a single functional equation is replaced by a system of linear functional equations for a system of Dirichlet series. In addition congruence conditions are stated for the basis numbers n of $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. This idea conduces to the statement that for example there is an infinity of such nontrivial functional equations between $\zeta(s)$ and the L-series. It seems inevitable for a discussion of $\zeta(s)$ to use also these relations being let beside till now, except in the additiv theory of prime numbers of Hardy-Littlewood by considering $\sum_p x^p$, where one has used similar statements. In this way one obtains in all important cases (for $\lambda > 2$) the finiteness of the number of solutions of these problems. This will be developed systematically later.

2) Furthermore, in order to specify in a linear set of solutions the most important single function, it is necessary (in the case of $\mathfrak{G}(1)$ = modular group) to notice a new, general arithmetical property of the power series of the modular functions. This is equivalent with the Euler product. For the simplest cases, the final results in terms of Dirichlet series are here appended.

We consider the complete set of linearly independent Dirichlet series with the same signature $(1, k, \gamma)$. Denote them by $\varphi^{(1)}(s)$, $\varphi^{(2)}(s), \dots \varphi^{(\kappa)}(s)$; each is associated with the modular group $\mathfrak{G}(1)$. To this we apply the recently developed theorems on modular functions.

Putting $\varphi^{(\nu)}(s) = \sum_{n=1}^{\infty} a^{(\nu)}(n) n^{-s}$, there exists a set of κ quadratic matrices $B^{(\nu)}$ with constant elements and of degree κ such that: The $B^{(\nu)}$ form a basis of a commutative ring of quadratic matrices of degree κ. The ring is of rank κ.

The matrices

$$(4) \qquad \lambda(n) = \sum_{\nu=1}^{\kappa} a^{(\nu)}(n) B^{(\nu)} = (\lambda_{\varrho\sigma}(n))$$

formed with the coefficients $a_n^{(\nu)}$ have a simple multiplication theorem

$$(5) \qquad \lambda(m) \cdot \lambda(n) = \sum_{d \mid m, n} \lambda\left(\frac{mn}{d^2}\right) d^{k-1}$$

and this property of the $\lambda(n)$ is equivalent to the assertion that the matrix of Dirichlet series $\Phi(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s} = \sum_{\nu=1}^{\kappa} \varphi^{(\nu)}(s) B^{(\nu)} = (\varphi_{\varrho\sigma}(s))$ has an Euler product $\Phi(s) = \prod_p (E - \lambda(p) p^{-s} + p^{k-1-2s} E)^{-1}$, $E = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Although it is not generally true that each $\varphi^{(\nu)}(s)$ has an Euler product, the set of functions $\varphi^{(\nu)}(s)$ can be joined by the matrices B^{ν} to a matrix $\Phi(s)$ having an Euler product.

Thus the coefficients $a^{\nu}(n)$ are known if the $a^{(\nu)}(p)$ are known only for the prime numbers p.

Now by using an arbitrary constant matrix A, $\det A \neq 0$,

$$\Phi^*(s) = A \cdot \Phi(s) \cdot A^{-1}$$

has also such an Euler product with $\lambda(p)$ replaced by $A \cdot \lambda(p) \cdot A^{-1}$. This is effected by replacing the basis $(\varphi^{(\nu)}(s))$ by another basis $A(\varphi^{(\nu)}(s))$.

But from a general lemma of algebra, it is possible to choose A in such a way that the commutative matrices $B^{(\nu)}$ are simultaneously brought in the triangular form

$$B^{\nu *} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1\kappa} \\ 0 & b_{22} & \cdots & b_{2\kappa} \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \cdots 0 & & b_{\kappa\kappa} \end{pmatrix}$$

with elements below the main diagonal being zero. Then the matrices $\Phi^*(s)$ and

$$\lambda^*(n) = A \cdot \lambda(n) \cdot A^{-1}$$

are also of this form. But the elements in the diagonal of $\Phi^*(s)$ are the characteristic roots of Φ^*, hence of Φ. Hence these characteristic roots belong to the linear set of $(\varphi^{(1)}(s), \ldots, \varphi^{(\kappa)}(s))$. They are the really interesting solutions of the $(1, k, \gamma)$-problem.