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Mike West Jeff Harrison

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PREFACE

In this book we are concerned with Bayesian learning and forecasting in dynamic environments. We describe the structure and theory of classes of dynamic models, and their uses in Bayesian forecasting.

The principles, models and methods of Bayesian forecasting have been developed extensively during the last twenty years. This development has involved thorough investigation of mathematical and statistical aspects of forecasting models and related techniques. With this has come experience with application in a variety of areas in commercial and industrial, scientific and socio-economic fields. Indeed much of the technical development has been driven by the needs of forecasting practitioners. As a result, there now exists a relatively complete statistical and mathematical framework, although much of this is either not properly documented or not easily accessible. Our primary goals in writing this book have been to present our view of this approach to modelling and forecasting, and to provide a reasonably complete text for advanced university students and research workers.

The text is primarily intended for advanced undergraduate and postgraduate students in statistics and mathematics. In line with this objective we present thorough discussion of mathematical and statistical features of Bayesian analyses of dynamic models, with illustrations, examples and exercises in each Chapter. On the less mathematical side, we have attempted to include sufficient in the way of practical problems, motivation, modelling and data analysis in order that the ideas and techniques of Bayesian forecasting be accessible to students, research workers and practitioners in business, economic and scientific disciplines.

Prerequisites for the technical material in the book include a knowledge of undergraduate calculus and linear algebra, and a working knowledge of probability and statistics such as provided in first and second year undergraduate statistics programs. The exercises are a mixture of drill, mathematical and statistical calculations, generalisations of text material and more practically orientated problems that will involve the use of computers and access to software. It is fair to say that much insight into the practical issues of model construction and usage can be gained by students involved in writing their own software, at least for the simpler models. Computer demonstrations, particularly using graphical displays, and use of suitable

software by students, should be an integral part of any university course on advanced statistical modelling and forecasting. Two microcomputer software packages, interactive, menu-driven and highly graphically based, have been developed and written over the last two or three years by the authors. The first, **FAB**,† is specifically designed for teaching and training in Bayesian forecasting, concentrating on models and methods contained in the first half of the book. The second, **BATS**,‡ is more orientated towards application and may be usfully used in gaining experience in the use of an important class of models, and associated intervention and monitoring techniques. Both have been used in teaching and training of students from a variety of backgrounds, and also of practitioners, and may be usefully studied in connection with the book.

The material in the book can be very loosely grouped into Chapters taken three at a time. There are sixteen Chapters, so consider five groups of three with the final Chapter 16 comprising a summary Appendix of mathematical and statistical theory relevant to the first fifteen.

A. Introduction

The first three Chapters provide a broad introduction to the basic principles, modelling ideas and practice of Bayesian forecasting and dynamic models. In Chapter 1 we discuss general principles of modelling, learning and forecasting, aspects of the role of forecasters within decision systems, and introduce basic elements of dynamic modelling and Bayesian forecasting. Chapter 2 is devoted to the simplest, and most widely used, dynamic model — the first-order polynomial model, or steady model. In this setting, the simplest mathematical framework, we introduce the approach to sequential learning and forecasting, describe important theoretical model features, consider practical issues of model choice and intervention, and relate the approach to well-known alternatives. Chapter 3 continues the introduction to dynamic modelling through simple dynamic

†Harrison, West and Pole (1987). **FAB**, a training package for Bayesian forecasting, *Warwick Research Report* **122**, Department of Statistics, University of Warwick

‡West, Harrison and Pole, (1987). **BATS**: Bayesian Analysis of Time Series, *The Professional Statistician* **6**, (43—46)

regression models. Readers will be familiar with standard regression concepts, so that the rather simple extension of straight line regression models to dynamic regression will be easily appreciated.

B. Dynamic linear model theory and structure

Chapters 4,5 and 6 provide a comprehensive coverage of the theoretical structure of the class of Dynamic Linear Models (DLMs) and Bayesian analyses within the class. Chapter 4 is key. Here we introduce the general framework and notation, and derive the major theoretical results for learning and forecasting. Chapter 5 is concerned with a special subclass, referred to as Time Series Models, that relate naturally to most existing methods for time series forecasting. Chapter 6 focusses on aspects of model design and specification, developing, in particular, the concepts of models built up from basic components, and discounting.

C. Classes of dynamic models

Chapters 7, 8 and 9 describe in greater detail the structure of important special classes of dynamic models, and their analyses. Chapter 7 is devoted to Time Series Models for polynomial trends, particularly important cases being first-order polynomials of Chapter 2, and second-order polynomials, or linear trend models. Chapter 8 concerns dynamic linear models for seasonal time series, describing approaches through seasonal factor representations and harmonic models based on Fourier representations. Chapter 9 concerns relationships between time series modelled through dynamic regressions, extending Chapter 3, models for transfer effects of independent variables, and dynamic linear model representations of classical ARIMA type noise models.

D. DLMs in practice, intervention and monitoring

Chapter 10 illustrates the application of standard classes of dynamic models for analysis and forecasting of time series with polynomial trends, seasonal and regression components. Also discussed are various practical model modifications and data analytic considerations. Chapter 11 focusses on intervention as a key feature of complete forecasting systems. We describe modes of subjective intervention in dynamic models, concepts and techniques of forecast model monitoring and assessment, and methods of feed-forward and feed-back

control. Chapter 12 is concerned with multi-process models by which a forecaster may combine several basic DLMs together for a variety of purposes. These include model identification, approximation of more complex models, and modelling of highly irregular behaviour in time series, such as outlying observations and abrupt changes in pattern.

E. Advanced topics

Chapters 13, 14 and 15 are concerned with more advanced and recently developed models. In Chapters 13 and 14 we consider approaches to learning and forecasting in dynamic, non-linear models, where the neat theory of linear models does not directly apply. Chapter 13 describes some standard methods of analytic and numerical approximations, and also some more advanced approaches based on numerical integration. Chapter 14 demonstrates analyses in the class of dynamic generalised linear models. In Chapter 15, we return to linear models but consider aspects of modelling and forecasting in multivariate settings.

Acknowledgements

Section 1.4 of Chapter 1 briefly reviews historical developments and influences on our own work. Over the years many people and organisations have contributed, directly and indirectly, to material presented in this book and to our own efforts in developing the material. In particular, we would mention Colin Stevens, Roy Johnson, Alan Scott, Mike Quinn, Jim Smith and our past research students Mike Green, Rei Souza, Jamal Ameen, Helio Migon, Muhammed Akram, Tomek Brus, Dani Gamerman and Jose Quintana. Special thanks go to Andy Pole, our SERC Research Fellow and colleague over the last three years. In addition to joint work on non-linear models, Andy's computer experience has contributed a great deal to the development of computer software which has been used for many of the examples and case studies presented in the book.

Among the companies who have supported the development of Bayesian forecasting we must single out Imperial Chemical Industries plc. The initial work on forecasting began there in 1957 and took

its Bayesian flavour in the late 1960's. We would particularly like to acknowledge the help and encouragement of Richard Munton, Steve Smith and Mike Taylor of ICI. Other companies have supported our work and aided software development. Among these are British Gas Corporation, through the guiding efforts of Chris Burston and Paul Smith, Information Services International (formerly Mars Group Services), and IBM.

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Mike West & Jeff Harrison
January 1989

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