STUDIES IN MATHEMATICS AND ITS APPLICATIONS

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12

APPLICATIONS OF VARIATIONAL INEQUALITIES IN STOCHASTIC CONTROL

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NORTH-HOLLAND

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English version edited, prepared and produced by
TRANS-INTER-SCIENTIA
P.O. Box 16, Tonbridge, TN11 8DY, England

1982

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM • NEW YORK • OXFORD

ISBN 0 444 86358 3

Translation of:
Applications des Inéquations Variationnelles
en Contrôle Stochastique
© Bordas (Dunod), Paris, 1978

Publishers:
NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM • NEW YORK • OXFORD

Sole distributors for the U.S.A. and Canada:
ELSEVIER NORTH-HOLLAND, INC.
52 VANDERBILT AVENUE, NEW YORK, N.Y. 10017

Library of Congress Cataloging in Publication Data

Bensoussan, Alain.
Applications of variational inequalities in stochastic control.

(Studies in mathematics and its applications; v. 12)
Translation of Applications des inéquations variationnelles en contrôle stochastique.
Bibliography: p.
1. Control theory. 2. Stochastic processes. 3. Differential equations, Partial. 4. Calculus of variations. 5. Inequalities (Mathematics) I. Lions, Jacques Louis. II. Title. III. Series.
QA402.3B4313 629.8'312 81-22404
ISBN 0-444-86358-3 AACR2

PRINTED IN THE NETHERLANDS

Applications of Variational Inequalities in Stochastic Control

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VOLUME 12

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FOREWORD

This book treats second order partial differential equations and unilateral problems, as well as stochastic control and optimal stopping-time problems. It deals with branches of mathematics which may at first sight appear totally different and which have developed along quite independent lines, but which are in fact strongly inter-related and which are capable of cross-fertilising each other. The fundamental link lies in the interpretation of the solutions of certain partial differential equations. This interpretation is an extension of the method of characteristics which allows the solution of a linear first-order hyperbolic equation to be expressed explicitly as a functional defined along the characteristic trajectories. A similar phenomenon arises in the case of parabolic or elliptic equations, but the characteristic trajectories then become stochastic processes. In very general terms, it is absolutely necessary to resort to probabilistic models if we wish to be able to give explicit formulas for the solutions of partial differential equations (or of systems of such equations).

With regard to nonlinear equations, an important method (but not the only one) for expressing the solution of these equations consists of using the techniques of optimal control. Again, this forms an extension of the Hamilton-Jacobi method in the calculus of variations. The Hamilton-Jacobi equation is a nonlinear hyperbolic equation of first order. Stochastic control leads to quasilinear equations. The book by Fleming-Rishel gives an excellent discussion of the state of the art. Certain variational inequalities which likewise constitute nonlinear problems also possess a probabilistic interpretation. In this case we are dealing with control problems in which the decision variable is a stopping time.

Chapter I, which is designed as an extended introduction, presents the problems in formal manner and gives a more detailed description of the contents of the book. We hasten to emphasise at this point that this book is by no means intended to be exhaustive in its treatment, either with respect to the probabilistic models used or to the control problems treated. The probabilistic models are limited to diffusions. The control can take effect via the drift or via the diffusion term, or it can even be a *stopping time*. We also investigate differential games problems, with or without stopping times.

Other probabilistic models and other control problems will be considered in a second volume. In particular, we shall treat impulse control, which leads to quasi-variational inequalities.

The book is designed so as to allow it to be read equally well by analysts and by probabilists, and we have followed a policy of using the formalism and the techniques from both disciplines. It is informative to be able to give, when possible, two proofs of a single result: an analytic proof and a probabilistic proof. We have endeavoured to do this in order to bring out the advantage of using the two types of approach in conjunction. The probabilistic methods undoubtedly are the more intuitive, in that in some circumstances they allow explicit formulas to be used for certain quantities. The analytic methods, on the other hand, are undoubtedly the more powerful and more elegant when the variational formulation and energy techniques can be applied. In this case, they are clearly more economical as far as assumptions are concerned. The probabilistic methods are very well suited to estimates in the space L^∞, and the analytic methods to estimates in Sobolev spaces.

However, our objective in the present book is not to investigate nonlinear problems of partial differential equations; rather, it is to obtain constructive methods which will allow us to calculate, if necessary by using the resources of Numerical Analysis, the solution of optimal control problems, in particular those with stopping times (and, in the second volume, with impulse controls). We have not attempted to take the subject matter as far as it can be taken, and we refer the reader to the bibliography for further developments (using similar methods); numerous applications are described in the references cited in the bibliography; in particular the reader may consult Goursat [1], Leguay [1], Maurin [1], Quadrat [1] and Robin [1]. For the *numerical aspects* we refer to Quadrat [1], [2], Quadrat and Viot [1], and Kushner [1] and, for the numerical solution of variational inequalities to R. Glowinski, J.L. Lions and R. Trémolières [1].

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CHAPTER 1

GENERAL INTRODUCTION TO OPTIMAL STOPPING TIME PROBLEMS

1. SYNOPSIS

This first chapter is intended to give a general introduction to the book as a whole, and does not by any means attempt to present a rigorous theory; we introduce stopping time problems in as intuitive a manner as possible, and we give a number of examples of application as well as some idea of the techniques which will be used and developed in the later chapters.

2. FORMAL DESCRIPTION OF STOPPING TIME PROBLEMS

We shall now give a description of the basic problems, initially taken to be as simple as possible. We shall discuss a number of extensions and more complicated situations a little later on.

We consider a *stochastic dynamic system*, whose *state* $y(t)$ ($\in R^n$) evolves in accordance with the following differential equation (in the sense of Ito):

$$(2.1)\qquad \begin{cases} dy = g(y(t))dt + \sigma(y(t))dw(t) \\ y(0) = x\ . \end{cases}$$

In (2.1), $g(x)$ and $\sigma(x)$ are given functions on R^n, in (respectively) R^n and $\mathcal{L}(R^n;R^n)$. Furthermore, $w(t)$ is a standardised n-dimensional Wiener process; i.e. we have

$$(2.2)\qquad \begin{cases} \forall t,\ w(t) \text{ is a Gaussian random variable with values in } R^n, \\ \text{with zero mean and with variance} \\ E\, w_i(t)\, w_j(s) = \delta_{ij} \min(t,s)\ ;\ i,j = 1 \ldots n. \end{cases}$$

The function g is termed the *drift* and the function σ is termed the *diffusion*. The initial state is $x \in R^n$, (in general non-random).

If $\sigma = 0$, (2.1) is an ordinary differential equation. Very formally, equation (2.1) states that if at the instant t the system has the known state $y(t)$, then over the interval $(t, t + \Delta t)$ (Δt small) the *variation* $\Delta y(t)$ of the state is a Gaussian R.V. (random variable) with mean $g(y(t))\Delta t$ and with variance $\sigma\sigma^*(y(t))\Delta t$.

Naturally, in order for (2.1) to be meaningful, it is necessary to make a number of assumptions with regard to the functions g and σ which ensure the existence and uniqueness (in a sense which will need to be defined) of the solution of (2.1).

We assume that we have access to all information on the past and present state of the system (*), (but not, of course, on the future state). The information at the instant t is then (mathematically) defined in terms of a σ-algebra $\mathfrak{F}^t$, such

(*) This will be the case in the majority of the situations considered in this book. Some cases in which only partial information is available will also be treated.

that $y(s)$ is $\mathcal{F}^t$- measurable for all $s \leq t$. The family $\mathcal{F}^t$ is an increasing family, which in practice can be either the family of σ-algebras generated by the process $y(t)$ itself,

$$\mathcal{F}^t = \sigma\text{-algebra generated by } y(s),\ s \leq t,$$

or the family generated by $w(t)$, or even a family with a wider definition.

The decision variable (the control !) is then a stopping time, i.e. a positive R.V. θ such that

$$\text{event } \{\theta \leq t\} \subset \mathcal{F}^t . \tag{2.3}$$

The property (2.3) means that at any instant t, taking account of the available information (i.e. $\mathcal{F}^t$), we know whether or not $\theta \leq t$.

Furthermore, let $\mathcal{O}$ denote a domain in R^n and let τ be the *first exit time* of the process $y(t)$ from $\mathcal{O}$, i.e.

$$\tau = \inf \{t \geq 0 \mid y(t) \notin \mathcal{O}\} \text{ (*)} .$$

We then define a cost function

$$\left\{ \begin{aligned} J_x(\theta) = E[\int_0^{\theta\wedge\tau} & f(y(t))(\exp\int_0^t c(y(s))ds)dt + \\ & + \psi(y(\theta))\chi_{\theta<\tau} \exp\int_0^\theta c(y)dt + \\ & + h(y(\tau))\chi_{\theta\geq\tau} \exp\int_0^\tau c(y)dt] \end{aligned} \right. \tag{2.4}$$

where the functions f, ψ, h, c, are given and where

$$\chi_{\theta<\tau} = \begin{cases} 1 & \text{if } \theta < \tau \\ 0 & \text{if } \theta \geq \tau \end{cases}$$

We observe that $y(t)$, τ depend on the point x, and this justifies using the notation $J_x(\theta)$ for the left-hand side of (2.4).

At this level x is a simple parameter, but it is of fundamental importance to introduce it explicitly for reasons which will become apparent later. The functional $J_x(\theta)$ decomposes into two parts, an *integral* cost, corresponding to what is paid while the process is not stopped, and a *final* cost which is itself expressible in two parts, namely $\psi(y(\theta))(\exp\int_0^\theta c\,dt)$ if $\theta < \tau$, (i.e. of we decide to stop the process before it exits from the domain $\mathcal{O}$), or $h(y(\tau))\exp\int_0^\tau c\,dt$ if $\theta \geq \tau$, (i.e. if the process is stopped after the instant at which it first exits from the domain $\mathcal{O}$).

Since it is permissible to take as unknown $\theta\wedge\tau = \min(\theta,\tau)$ instead of $\mathcal{O}$, we may interpret $\mathcal{O}$ as being a constraint on the stopping time, i.e. the process will be stopped, at the latest, when it first exits from $\mathcal{O}$; where this has not already been done, a balance will be established between the integral cost and the final cost. Naturally, $\mathcal{O}$ may be equal to R^n, in which case $\tau = +\infty$.

Finally, the exponential term may be interpreted as an *actualisation of the costs*; for example if $c(x) = -\beta$, where β is a positive constant, then β is a rate of interest.

(*) If $x \notin \mathcal{O}$, then $\tau = 0$.

We put

$$u(x) = \inf_{\theta} J_x(\theta). \tag{2.5}$$

The first fundamental problem is concerned with the analytic characterisation of the function u (is it possible to find a *set* of *relations* of which u is a solution, and if possible the unique solution?). Having obtained this analytic characterisation, the second fundamental problem is then to use this to deduce the existence of an optimal stopping time, i.e. a stopping time $\hat{\theta}_x$ such that

$$u(x) = J_x(\hat{\theta}_x) \quad \forall x . \tag{2.6}$$

Naturally, in addition to establishing the existence, it is important to be able to give qualitative information on θ_x and also to provide some means of calculating the solution.

The analytic characterisation of the function u proceeds via the definition of an *adequate functional* space, which poses the problem of the regularity of u as a function of x. ■

So far, we have been describing a *stationary* stopping time problem; i.e. the functions appearing in (2.1) and (2.4) do not depend on time, the initial instant is 0, the horizon is infinite (i.e. θ is not bounded above, a priori, by a number T).

We shall now give the "*nonstationary* analogue" of the above problem. The functions g, σ, f, ψ, h, c in this case depend on time. We also take a function $\bar{u}(x)$ and a horizon $T < \infty$. The initial instant is $t \le T$; the evolution of the system over $[t,T]$ is described by

$$\begin{aligned} dy &= g(y,s)ds + \sigma(y,s)dw(s), \quad s > t \\ y(t) &= x . \end{aligned} \tag{2.7}$$

If θ is a stopping time such that $\theta \in [t,T]$, we put

$$\left\{ \begin{aligned} J_{xt}(\theta) = E\Big[&\int_t^{\theta\Lambda\tau} f(y(s),s)\Big(\exp\int_t^s c(y,\lambda)d\lambda\Big)ds + \\ &+ \psi(y(\theta),\theta)\chi_{\theta<\tau\Lambda T} \exp\int_t^\theta c(y,s)ds + \\ &+ h(y(\tau),\tau)\chi_{\tau\le\theta,\tau<T} \exp\int_t^T c(y,s)ds + \\ &+ u(y(T))\chi_{T=\theta\Lambda\tau} \exp\int_t^T c(y,s)ds\Big]. \end{aligned} \right. \tag{2.8}$$

The stopping time τ is the first instant of exit from $\mathcal{O}$, after t.(*) The terms f, ψ, h have a meaning analogous to that given for (2.4). The supplementary term $u(y(T))\chi_{T=\theta\Lambda\tau}\exp\int_t^T c\,ds$ is the final cost when we decide not to stop the process before T and if the process has not left $\mathcal{O}$ before T.

The analogue of (2.5) becomes

(*) If $x \notin \mathcal{O}$, then $\tau = t$.

(2.9) $$u(x;t) = \operatorname*{Inf}_{\theta} J_{xt}(\theta) .$$

The problems which arise are essentially the same as in the stationary case.

3. ANALYTIC CHARACTERISATION BY DYNAMIC PROGRAMMING

Dynamic programming provides a (formal but highly intuitive) technique for obtaining the relations which the function u has to satisfy. For further generality, we shall consider the nonstationary case.

We put

$$Q = \mathcal{O} \times]0,T[\ , \ \Sigma = \partial \mathcal{O} \times]0,T[\ , \ \Gamma = \partial \mathcal{O} .$$

A certain number of relations are obvious. First,

(3.1) $$u(x,T) = \bar{u}(x), \ u(x,t) = h(x,t), \ x,t \in \Sigma .$$

Moreover, from the definition of u, we have

$$u(x,t) \leq J_{xt}(t)$$

and if $x \in \partial \mathcal{O} = \Gamma$ and $t < T$ we have $J_{xt}(t) = \psi(x,t)$, so that

(3.2) $$u(x,t) \leq \psi(x,t) \ , \ x \in \mathcal{O}, \ t < T.$$

Let us now consider that the stopping times θ have to satisfy the constraint

$$T \geq \theta \geq t + \delta \ , \ \delta > 0$$

where δ will tend to 0 (of course, this assumes $t < T$). We also assume $x \in \mathcal{O}$; then $\tau > t$ and for δ sufficiently small (random), we have $t + \delta \leq \tau$ (*). At the instant $t + \delta$, the state of the system has become (approximately)

$$x + \delta g(x,t) + \sigma(x,t)(w(\delta+t) - w(t)).$$

From the definition of the function u, we will have to 'pay', at the instant $t + \delta$, a cost greater than or equal to

$$u(x+\delta g(x,t) + \sigma(x,t)(w(\delta+t) - w(t)),t+\delta) ,$$

this cost having to be actualised at the instant t, and therefore multiplied by $\exp \int_t^{t+\delta} c ds$ where $\exp \int_t^{t+\delta} c ds \sim (1+\delta c(x,t))$. We must also pay the integral cost between t and $t + \delta$, the process not having been stopped on this interval. Thus, by forcing the process not to stop on the time interval $(t, t + \delta)$, we may expect to have to pay, at most, a cost (approximately) equal to

$$X = \delta f(x,t) + E(1+\delta c(x,t))u(x+\delta g(x,t) + \sigma(x,t)(w(\delta+t) - w(t)),t+\delta).$$

However, from the definition of the function u, $u(x,t) \leq X$.

(*) The formal manipulations which follow actually suppose δ to be non-random ...

If the function u is once-differentiable with respect to t and twice-differentiable with respect to x, we can write an expansion of X in terms of δ, as follows

$$X = \delta f + u + \delta c\, u + \delta \frac{\partial u}{\partial x} \cdot g + \delta \frac{\partial u}{\partial t} + E \frac{\partial u}{\partial x} \cdot \sigma(w(t+\delta) - w(t)) + $$
$$+ \frac{1}{2} E \frac{\partial^2 u}{\partial x^2} \sigma(w(t+\delta) - w(t)) . \sigma(w(t+\delta) - w(t)) .$$

From the properties of the Wiener process, this latter expectation is equal to (tr denoting the trace):

(3.4) $$\frac{1}{2} \operatorname{tr} \frac{\partial^2 u}{\partial x^2} \sigma\sigma^* \delta .$$

It is therefore of *first* order in δ and not of second order, as might have been expected by analogy with Taylor expansions in a deterministic case.

Moreover

(3.5) $$E \frac{\partial u}{\partial x} \cdot \sigma(w(t+\delta) - w(t)) = 0 .$$

We now reconsider the inequality $u \leq X$, taking account of (3.3), (3.4),(3.5). The term u (of order 0) vanishes. We can therefore divide by δ , and by making δ tend to 0 we obtain:

(3.6) $$f + cu + \frac{\partial u}{\partial x} \cdot g + \frac{\partial u}{\partial t} + \frac{1}{2} \operatorname{tr} \frac{\partial^2 u}{\partial x^2} \sigma\sigma^* \geq 0 .$$

We are thus led to introduce a family, indexed by t, of differential operators with respect to x, namely

(3.7) $$A(t) = - \sum_{ij} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_j g_j \frac{\partial}{\partial x_j} - c$$

where the matrix $a = a_{ij}$ is defined by

$$a = \frac{\sigma\sigma^*}{2} .$$

With this notation, (3.6) may be rewritten in the form

(3.8) $$- \frac{\partial u}{\partial t} + A(t) u \leq f .$$

Finally, it is worth noting that at the instant t we have two possibilities: either we stop the process immediately, or we allow the process to evolve freely