CLAYTON W. DODGE

the circular functions

CLAYTON W. DODGE
assistant professor of mathematics
university of maine

circular functions

PRENTICE-HALL, INC.
englewood cliffs, new jersey

PRENTICE-HALL MATHEMATICS SERIES

PRENTICE-HALL INTERNATIONAL, INC. *London*
PRENTICE-HALL OF AUSTRALIA, PTY., LTD. *Sydney*
PRENTICE-HALL OF CANADA, LTD. *Toronto*
PRENTICE-HALL OF INDIA (PRIVATE), LTD. *New Delhi*
PRENTICE-HALL OF JAPAN, INC. *Tokyo*

The Circular Functions

by Clayton W. Dodge

Current printing (last digit):

10 9 8 7 6 5 4 3

Library of Congress Catalog Card Number: 66-13040.
Printed in the United States of America. 13464-C.

TO MOM

in fond memory
of such delicacies
as peanut butter soup

preface

This text of twenty-seven sections has been designed for students preparing for the calculus, for teachers of such students, and for others interested in learning trigonometry. Its modern approach to trigonometry may be covered in from fifteen to forty-five class sessions with a design center of a two-hour, one-semester course. Thus each section should be covered in about one class hour.

Among the innovations in this text is the inclusion of a chapter on the history of trigonometry written by Howard Eves, along with problems in the inimitable Eves style.

The introduction of the term *angle* has been postponed until after the student has studied the basic properties of the circular functions as real functions of the real variable arc length on a unit circle. It is felt that the student will have a better appreciation of radian measure with this approach.

Because of the importance of sets and set notation, a brief appendix on that subject, which can be covered in one class hour, has been included and should be studied first of all whenever students have not already been exposed to this material.

Chapter 5 and/or Chapter 6 may be omitted in a shorter course.

In this text the *en quad* symbol ▌ is used to indicate the completion of a proof, and *iff* is read and means "if and only if."

Exercises whose results are needed later in the text, namely 7-3, 7-4, 10-2, 11-2, and 11-3, are starred so that they will not be omitted. Exercises numbered above 50 are considered supplementary—some are quite challenging—and no answers are given for them in the text.

My deep appreciation is extended to Prentice-Hall, Inc., for their generous help in preparing this text for publication. My gratitude also goes to Holt, Rinehart, and Winston, Inc., for their permission to quote the bulk of the material in Chapter 6 from Howard Eves' *An Introduction to the History of Mathematics*, Rev. ed. (1964). Especially I extend my heartfelt thanks to my good friend Howard Eves for countless kindnesses, and to my wife, Donna, for her patient understanding.

CLAYTON W. DODGE

Old Town, Maine

contents

CHAPTER ONE

preliminary analytic geometry

CHAPTER TWO

the circular functions

CHAPTER THREE

trigonometry

CHAPTER FOUR

complex numbers

CHAPTER FIVE

harmonic motion

table of symbols

The following symbols are defined on the pages indicated.

$A(a)$	rectangular coordinate	3
$\|x\|$	absolute value	4, 70
$\overline{AB}$	length of a segment	4, 6
$\overrightarrow{AB}$	directed length of a segment	4
$P(a, b)$	rectangular coordinates	5
▮	end of proof	(*footnote*)7
R^2	the set of points in the plane	9
$F(x)$	relation at x	10
F^{-1}	inverse relation	12
Δx	a change in x	14
$\doteq$	approximately equal	21
$P[\theta]$	circular coordinate	22
$\sin^2 \theta$, $\cos^4 \theta$, etc.	powers of the circular functions	33
$\sin^{-1}$, etc.	inverse circular relations	47
$\measuredangle$, $m(\measuredangle \overrightarrow{UOP})$	angle, measure of an angle	53
$^\circ$, $'$, $''$	degrees, minutes, seconds	55
$P[r, \theta]$	polar coordinates	69
$\in$, $\notin$	element	129
$\{ , \}$	set	130
;, |	such that	130
$\emptyset$	null set	130
$=$, $\neq$	equality of sets	131
$\subset$, $\not\subset$, $\supset$, $\not\supset$	subset	131
$\cap$	intersection	132
$\cup$	union	132
$'$	complement	133

CHAPTER SIX

history of trigonometry

appendices

the circular
functions

CHAPTER *1*

preliminary analytic geometry

ONE

linear analytic geometry

1.1 Definition

To each point on a (horizontal) line assign a real number as follows. First a point O, called the *origin*, is chosen to have the *coordinate* 0 (zero). Next choose a point U (for *unit*), generally to the right of O, to have the *coordinate* 1. Now any other point P is assigned a *coordinate* whose numerical value is the numerical value of the ratio $\overline{OP}/\overline{OU}$ and which is negative if O lies between P and U and is positive otherwise. Such a line is called a *number line*. (See Figure 1.) We write $A(a)$ to denote that point A has coordinate a.

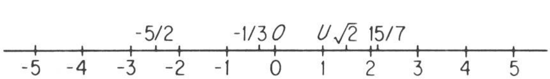

FIG. 1

We have thus set up a one-to-one correspondence between the real numbers and points on a line.

Remember that $|x| = x$ if $x \geqq 0$ and $|x| = -x$ if $x < 0$. Thus $|-2| = 2$, $|5| = 5$, etc.

1.2 Definition

The *distance between points* $A(a)$ *and* $B(b)$, denoted $\overline{AB}$, is defined by: $\overline{AB} = |b - a|$. The *directed distance from* $A(a)$ *to* $B(b)$, denoted $\overrightarrow{AB}$, is defined by: $\overrightarrow{AB} = b - a$.

Thus $\overrightarrow{AB} > 0$ if B is to the right of A and $\overrightarrow{AB} < 0$ if B is to the left of A. Note that $\overline{AB} = |\overrightarrow{AB}| = |\overrightarrow{BA}|$.

1.3 Examples

If $A(5)$ and $B(7)$, then $\overline{AB} = \overline{BA} = |7 - 5| = 2$. Also $\overrightarrow{AB} = 7 - 5 = +2$ whereas $\overrightarrow{BA} = 5 - 7 = -2$. If $C(-2)$ and $D(3)$, then $\overline{CD} = \overline{DC} = |3 - (-2)| = |5| = 5$, $\overrightarrow{CD} = +5$, and $\overrightarrow{DC} = -5$. If $E(-3)$ and $F(-5)$, then $\overrightarrow{EF} = (-5) - (-3) = -2$. (See Figure 2.)

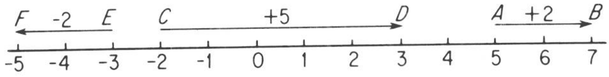

FIG. 2

exercises

1-1. Find $\overline{AB}$, $\overrightarrow{AB}$, and $\overrightarrow{BA}$ if $A(3)$ and $B(7)$.

1-2. Find $\overline{AB}$, $\overrightarrow{AB}$, and $\overrightarrow{BA}$ if $A(-2)$ and $B(3)$.

1-3. If $A(a)$ and $B(b)$, show that the midpoint M of segment AB has coordinate $(a + b)/2$.

1-4. If $A(2)$ and $B(6)$, find M, the midpoint of AB.

1-5. If $A(-3)$ and $B(5)$, find M, the midpoint of AB.

1-6. If A, B, and C are any three points on a line, show that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

1-7. If A, B, C, and D are any four points on a line, show that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = 0$.

1-8. Check the theorems in exercises 1-6 and 1-7 using $A(5)$, $B(2)$, $C(-3)$, and $D(-1)$.

1-9. Show that $|a|^2 = a^2$, whence $\sqrt{a^2} = |a|$.

TWO

plane analytic geometry

2.1 Definition

To each point P of the plane assign a pair (a, b) of *coordinates* as follows. Choose a pair of congruent number lines, called the *x-axis* and *y-axis*, intersecting at right angles at their origins, which point is called the *origin* O with *coordinates* $(0, 0)$. (Usually the x-axis is horizontal and positive to the right, the y-axis positive upward.) Points $A(a)$ on the x-axis and $B(b)$ on the y-axis are given *coordinates* $(a, 0)$ and $(0, b)$ respectively. To any other point P in the plane assign *coordinates* (a, b) where

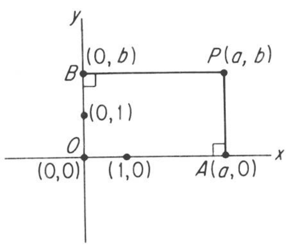

FIG. 3

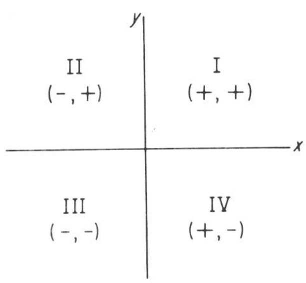

FIG. 4