LECTURES IN TOPOLOGY

THE UNIVERSITY OF MICHIGAN CONFERENCE OF 1940

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1941

FOREWORD

The present volume constitutes what may justifiably be called the "proceedings" of the Topology Conference which was held at the University of Michigan, June 24–July 6, 1940. According to the original plan, the Conference was to have been international in scope, with leaders in the field from several countries participating. Later it became evident that the changing world situation would make such participation impossible. Fortunately the development of topological research in this country has been of such a character that the resulting limitation in personnel did not necessitate restriction in the subject matter of the program. Therefore the publication in full herewith of the twelve principal lectures of the Conference, together with summaries of the shorter papers, may be regarded as a survey of the present status of topology in its various phases.

The editors wish to seize this opportunity to express, not only for themselves but on behalf of all those who attended the Conference, their grateful appreciation to the administrators of the Alexander Ziwet Fund whose generous financial assistance made the Conference possible; the Horace H. Rackham School of Graduate Studies of the University of Michigan and Dean Clarence S. Yoakum, not only for constant encouragement but for graciously welcoming the members of the Conference and providing comfortable physical facilities for the meetings; Director Louis A. Hopkins of the Summer Session for aid in arrangements; and the University of Michigan Press and, more specifically, Drs. Frank E. Robbins and Eugene S. McCartney for giving so generously of their editorial assistance and counsel.

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ABSTRACT COMPLEXES

By SOLOMON LEFSCHETZ

1. Introduction. The starting point of Poincaré, the true creator of the theory of complexes, was the assignment to pairs of faces of consecutive dimensions in a polyhedron $\prod$ of certain *incidence numbers* depending upon their geometric incidences and relative orientations. From the combinatorial systems thus arising he was able to extract certain fundamental integers, the *Betti numbers* and *torsion coefficients*, which turned out to be topological invariants of $\prod$ (surmise by Poincaré, proof by Alexander and Veblen).

In Poincaré's own work complexes were always of the more restricted type—the *manifolds*. Furthermore, for him a complex was always a polyhedron. Endeavors followed, however, to obtain an abstract system with more or less similar properties. Thus there arose different kinds of *abstract complexes*, and they fall essentially into three categories:

A. *Abstract complexes in the sense of Dehn-Heegaard* (*related types of M. H. A. Newman and J. W. Alexander*). The motivation is found in the following still unsolved question, which goes back (in some form) to Poincaré: When two polyhedra $\prod, \prod'$ are topologically equivalent do they or do they not have isomorphic subdivisions? This brings to the fore the operation of *subdivision* and similar operations on the faces. In this type of complex, then, certain a priori operations of this nature are postulated, and the complexes are investigated with respect to equivalence under them. The theory thus obtained is strongly "loaded" on the geometric side, and for that reason has not as yet been very widely developed.

B. *Abstract complexes in the sense of W. Mayer.* Here we find the opposite extreme; only the purely group-theoretic features are preserved. This type has been systematically developed (up to a point) by its originator, and perhaps deserves much fuller attention. A variant has recently been considered by A. W. Tucker, who reduced a (finite) complex to a pair of matrices subjected to certain simple relations.

C. *Abstract complexes in the sense of A. W. Tucker.* They occupy a middle ground between A and B in that their structure partakes of algebra and also of geometry, which enters in through certain order relations. This type appears to be best adapted to the applications of present-day topology, and so it alone will be dealt with here.

Henceforth we shall drop the term "abstract" and merely speak of "complexes"; it is understood that they are the structures of category C, to be defined presently.

2. Definition of complexes. A complex is a set $X = \{x\}$ (by no means always finite) whose elements are ordered (i.e. partially ordered) with respect to a proper reflexive and transitive relation $<$ (is a face of) and with two attached integral-valued functions of the elements x and of their pairs x, x':

the *dimension* of x, written $\dim x$;

and the *incidence number* of x, x', written $[x : x']$, under the following conditions:

I. $x' < x$ implies $\dim x' \leqq \dim x$;

II. $[x : x'] = [x' : x]$;

III. $[x : x'] \neq 0$ implies $x < x'$ or $x' < x$ and also $|\dim x - \dim x'| = 1$;

IV. Given x, x'' such that $|\dim x - \dim x''| = 2$, there are at most a finite number of x' such that $[x : x'][x' : x''] \neq 0$ and $\sum_{x'} [x : x'][x' : x''] = 0$.

The dimension of X itself, written $\dim X$, is the largest number dim x for $x \varepsilon X$. If dim $X = n$ is finite, X is sometimes called an *n-complex.*

As a matter of fact, a certain latitude must be allowed in the incidence numbers. Let $\alpha(x)$ be a function whose value is ± 1. Then if X, with the assigned numbers $[x : x']$, is a complex, it remains one when they are replaced by $[x : x']_1 = \alpha(x)\alpha(x')[x : x']$. We agree to consider the new complex the same as the old. Thus the numbers $[x : x']$ are defined only up to a factor $\alpha(x)\alpha(x')$. The function $\alpha(x)$ is known as an *orientation function*; the passage from $[x : x']$ to $[x : x']_1$ is described as *reorienting* X.

Isomorphism, weak isomorphism. If $X = \{x\}$, $Y = \{y\}$ are two complexes and there is a one-one correspondence $x \leftrightarrow y$ preserving $<$, dim x, $[x : x']$ (up to a product $\alpha(x)\alpha(x')$), the transformation $t : X \to Y$ is said to be *isomorphic.* If the situation is the same but dim $y =$ dim $x + k$, then t is said to be *weakly isomorphic.*

Let us introduce for each x a new element x^* and introduce in $X^* = \{x^*\}$ the following relations: $x^* < x'^*$ whenever $x' < x$; dim $x^* = -$ dim x; $[x^* : x'^*] = [x : x']$. The basic conditions I–IV being still verified, X^* is a complex, the *dual* of X. We find at once $X^{**} \cong X$, so that X, X^* are $\cong$ the dual of one another. Notice that the symmetry in the relationship has been made possible only through the admission of *negative* dimensions.

Suppose that X is n-dimensional and contains no negative dimensional elements. Let $\overline{X} = \{\bar{x}\}$ be a weak isomorph of X^* with dimensions raised by n. The resulting $\overline{X}$, the *reciprocal* of X, is a complex whose dimension $\leqq n$ and likewise such that dim $\bar{x} \geqq 0$. In many cases $\overline{X}$ may advantageously replace X^*. In the earlier theory of manifolds it was to $\overline{X}$ that the term "dual" was always applied.

Notations. The elements of X will be written $x^p{}_i$, where $p =$ dim x. The elements $(x^p{}_i)^*$ of X^* will be written $x^i{}_p$. Thus if the dimensional index p is a subscript the dimension is $- p$.

The summation convention relative to repeated non-dimensional indices will be adopted throughout.

Star, closure, subcomplexes. Returning to the initial complex X, we have two fundamental functions of the elements:

St x, the *star* of x, defined by $\mathrm{St}\, x = \{x' | x < x'\}$;
Cl x, the *closure* of x, defined by $\mathrm{Cl}\, x = \{x' | x' < x\}$.

Another convenient function is $\mathcal{B}x$, the boundary of x, defined by $\mathcal{B}\, x = \mathrm{Cl}\, x - x$.

A complex X is *star-finite* (*closure-finite*) whenever every St x (Cl x) is finite. When X is both star- and closure-finite it is said to be *locally finite.* These are actually the only three types of infinite complexes that occur in the applications (probably because in the absence of suitable methods of attack the others are studiously avoided).

The functions St, Cl are also defined for a subset X_0 of X; St X_0 (Cl X_0) is the union of the stars (closures) of all the elements of X_0.

Subcomplexes. A subcomplex Y of X is a subset which under the order relations, dimensions, and incidence numbers already existing in X, is also a complex. We say that Y is a *closed* subcomplex if $\mathrm{Cl}\, Y = Y$; that it is an *open* subcomplex if $\mathrm{St}\, Y = Y$. If Y is a closed (open) subcomplex of X, then Y^* is an open (closed) subcomplex of X^*. The effect of this dualism will be perceived later.

Before going further it will be profitable to examine some of the more important types of complexes occurring in topology.

3. Simplicial complexes. A *p-simplex* σ^p is any set of $p + 1$ objects $\{A_0, \cdots, A_p\}$. The A_i are the *vertices of* σ^p. The $\sigma^q \subset \sigma^p$ are the *q-faces* of σ^p (proper faces when $q < p$). If the vertices of σ^q and σ^r are distinct, and if together they comprise those of σ^p, we will write $\sigma^p = \sigma^q\sigma^r$. To *orient* σ^p is to assign to its vertices an order modulo an even permutation. If the order is

$A_0, \cdots, A_p$ we also write $\sigma^p = A_0 \cdots A_p$. Suppose that σ^p, σ^{p-1} ordered are such that with suitably labeled vertices $\sigma^p = A_0\sigma^{p-1}$. We then say that σ^p, σ^{p-1} are *positively related*; otherwise that they are *negatively related*. The *dimension* of σ^p, written dim σ^p, is p. The set of proper faces of σ^p, written $\mathcal{B}\sigma^p$, is the *boundary* of σ^p.

A *simplicial* complex is a complex K whose elements $\{\sigma\}$ are simplexes such that if $\sigma \,\varepsilon\, K$ every face of σ also $\varepsilon\, K$ and, moreover: (a) $\prec$ means "is a face of"; (b) dim $\sigma^p = p$; (c) if $\sigma^{p-1} \prec \sigma^p$ and they are positively related, then $[\sigma^p : \sigma^{p-1}] = [\sigma^{p-1} : \sigma^p] = 1$; if they are negatively related the same incidence numbers are -1; all the other $[\quad : \quad]$ are zero. Conditions I, II, III hold, and IV is readily verified. K is closure-finite.

A closed subcomplex of K is a simplicial complex; an open subcomplex of K is a complex known as an *open* simplicial complex (it may be defined directly without the use of K). By contrast a simplicial complex is sometimes called a *closed simplicial* complex. As an example, σ^p together with all its faces is a closed simplicial complex, written $\bar{\sigma}^p$; similarly $\mathcal{B}\sigma^p$; hence $\sigma^p = \bar{\sigma}^p - \mathcal{B}\sigma^p$ is an open simplicial complex.

Derived complex. Given any complex (not necessarily simplicial) $X = \{x\}$, take as simplexes all the subsets $x' = x_1 \cdots x_r$ such that $x_1 \prec \cdots \prec x_r$. Then $X' = \{x'\}$ is a simplicial complex known as the *first derived* of X. By repetition we have the successive derived $X'', \cdots, X^{(p)}, \cdots$.

Euclidean complex. Let $K = \{\sigma\}$ be as before and let $\{A_\alpha\}$ be its vertices (the σ^0). Choose for each A_α a real parameter $x_\alpha : 0 \leqq x_\alpha \leqq 1$. Define now a point x as a set $\{x_\alpha\}$ such that: (a) all but a finite number of the x_α are zero; (b) $\sum x_\alpha = 1$; (c) if $x_{\alpha_0}, \cdots, x_{\alpha_p}$ are the $x_\alpha \neq 0$, then $\sigma^p = A_{\alpha_0} \cdots A_{\alpha_p} \,\varepsilon\, K$. Adopt for $\{x\}$ a euclidean metric. The aggregate of all points under (c) is called a *euclidean p-simplex*, written σ_e^p. The order, dimension, and incidence relations existing in K are extended to $K_e = \{\sigma_e^p\}$.

It is thus a metric space partitioned into euclidean simplexes making up a complex isomorphic with K. We call K_e a *euclidean complex.* The term is then extended to a collection $K_{1e} = \{\sigma_{1e}\}$ such that there exists a topological mapping $T: K_e \rightarrow K_{1e}$, which is affine on each σ_e.

Let $\hat{\sigma}_e$ be a point of σ_e. Construct an isomorph K_e' of the first derived, K', such that $\hat{\sigma}_e$ corresponds to the vertex σ of K'. There results a decomposition of K_e into a new euclidean complex K_e' called a *derived* of K_e. Similarly, one may obtain K_e'', $\cdots$. Generally the $\hat{\sigma}_e$ are chosen as the centroids of the σ_e, and the derived are then said to be *barycentric.*

Geometric complex. Let K_e be locally finite and let T be a topological transformation applied to K_e. The $T\sigma_e = \zeta$ are called *geometric simplexes.* The "K_e relations" are extended to $\mathcal{K} = \{\zeta\}$, and the complex thus arising (still isomorphic with K and also with K_e) is said to be a *geometric complex.*

With the $\mathcal{K}$ type we already have access to more colorful examples. For any "smooth" surface $\sum$ decomposed into curvilinear triangles such that no two triangles or edges have the same vertices is easily identified with a two-dimensional $\mathcal{K}$. The surface $\sum$ need not be closed, and may have a boundary. The one-dimensional geometric complexes are known as *graphs*; it is with graphs that the earliest topological research was chiefly concerned.

In topology an important rôle is played by a complex introduced by Vietoris and made the basis of his homology theory. The complex K consists of all the simplexes whose vertices are the points of a compact metric space $\mathcal{R}$. It is closure-finite, but is not star-finite unless the space $\mathcal{R}$ reduces to a finite set. Noteworthy also are the *regular* complexes which we have found, for instance, convenient in connection with local connectedness. Such a complex is a countable locally finite subcomplex $L = \{\sigma_n\}$ of K such that diam $\sigma_n \rightarrow 0$; L may also be regular with respect to a given closed and bounded subset A of $\mathcal{R}$; then

$\sup\{\operatorname{diam}\sigma_n, d(\sigma_n, A)\} \to 0$. These complexes have received interesting recent applications at the hands of Steenrod (see his lecture, pp. 43–55) and Eilenberg.

4. Polyhedra. It will be remembered that an n-cell is a point for $n = 0$, and for $n > 0$ the topological image of the euclidean region $x_1^2 + \cdots + x_n^2 < 1$. A *convex* polyhedral n-cell ω^n is a point for $n = 0$, and for $n > 0$ a bounded convex region in a euclidean n-space $\mathcal{E}^n$, whose boundary consists of a finite set of ω^r, $r < n$. A *polyhedron* $\prod$ is a metric space partitioned into disjoint convex polyhedral cells whose set $\{\omega_i^n\}$ has the property that $\bar{\omega}_i^n - \omega_i^n$ is a finite number of ω^p, $p < n$, which are also in the set. The polyhedron will now be made a complex, as follows: The order relation, $\prec$, signifies "is a face of"; and $\dim \omega_i^n = n$. The incidence numbers $[\omega_i^n : \omega_j^{n-1}]$ are determined thus: In the space $\mathcal{E}_i^n$ of ω_i^n choose a system of coördinates $x_1^i, \cdots, x_n^i$. Then, if ω_j^{n-1} is not a face of ω_i^n, set $\{\omega_i^n : \omega_j^{n-1}\} = 0$. If $\omega_j^{n-1} \prec \omega_i^n$, apply a euclidean transformation of coördinates in $\mathcal{E}_i^n$, bringing the first $n - 1$ coördinates in coincidence with $x_1^j, \cdots, x_{n-1}^j$, and the last so that ω_i^n is in the region $x_n^i > 0$. If $\alpha\,(= \pm 1)$ is the determinant of the transformation we define $[\omega_i^n : \omega_j^{n-1}] = \alpha$. The rest of the $[\quad : \quad]$ are determined to accord with II, III. Thus I, II, III are fulfilled, IV is readily verified, and so $\prod$ is a complex.

The process utilized for constructing an isomorph of the derived of euclidean complexes may be applied to $\prod$. It is, in fact, the euclidean complexes thus arising which are usually meant by the derived $\prod'$, $\prod''$, etc. Furthermore, in general they are taken as barycentric (in the obvious sense).

We need not mention the familiar polyhedra of elementary geometry. A noteworthy polyhedron arises as follows: Let A be a closed and bounded subset of the euclidean space $\mathcal{E}^n$ and consider the reticulation R_k cut out by all the subspaces $x_i = m_i 2^{-k}$, where the m_i take the values $0, \pm 1, \cdots$. Let $\{\omega_{ki}\}$ be the elements of R_k and S_k be the set of all the ω_{ki} whose closures meet A. The

$\omega_{k+1,i} \subset S_k$ make up a set S_k'. Then the union of $R_0 - S_0$, $S_0' - S_1$, $S_1' - S_2 \cdots$ is a polyhedron $\prod$ of dimension n which covers $\mathcal{E}^n - A$ and which may be utilized to great advantage in topology. Of course, $\prod'$ is a euclidean complex fulfilling the same rôle. Notice that $\prod, \prod'$ are regular with respect to A in a sense analogous to that of No. 3: if $\prod = \{\omega_p\}$, then sup $\{\text{diam } \omega_p, d(A, \omega_p)\} \to 0$.

5. Cellular complexes. These complexes, when finite, were investigated by Veblen. The description of a cellular complex $K = \{E_i^n\}$ is the same as for a polyhedron, save that the cells are unrestricted and, in addition, $\overline{E}_i^n$ is topologically equivalent to a euclidean closed spherical region, $x_1^2 + \cdots + x_n^2 \leq 1$, and this under a topological mapping t_i^n which sends $\overline{E}_i^n - E_i^n$ into the sphere $S_i^{n-1} : x_1^2 + \cdots + x_n^2 = 1$. A definite mapping is chosen for each E, and a "relative orientation" of E_i^n and $E_j^{n-1} < E_i^n$ selected by reference to the images in $\mathcal{E}^n$, after a manner more or less patterned upon the process for a polyhedron. The relative orientation may be characterized by ± 1, and thus serve to determine the incidence numbers. The ordering and dimensions are introduced as before, and I–IV are readily verified. Thus K is a complex in the sense of our definition.

6. Products of complexes. Products are devices for constructing new complexes out of given complexes. It will be sufficient to consider the product of two complexes: $X = \{x\}$, $Y = \{y\}$. The elements of the new complex are the pairs (x, y) of the cartesian product of the two sets of elements and are designated by $x \times y$. The order relations are governed by $x \times y < x' \times y'$ whenever $x < x'$ and $y < y'$. We set $\dim x \times y = \dim x + \dim y$, and finally define

$$[x \times y : x' \times y] = [x : x'],$$
$$[x \times y : x \times y'] = (-1)^{\dim x}[y : y'],$$

which, by imposing No. 2, II, III, suffices to determine all the incidence numbers. The verification of No. 2, IV is elementary, and so $\{x \times y\}$ is a complex designated by $X \times Y$, the *product* of X and Y. Many properties are quickly derived from the definition. We merely state:

I. $(X \times Y)^* \cong X^* \times Y^*$;

II. When $X \cap Y = 0$, $Y \times X$ is merely $X \times Y$ reoriented by the factor $\alpha(x \times y) = (-1)^{\dim x \dim y}$;

III. The product $K \times L$ of two simplicial complexes K, L is *not* a simplicial complex. The product $\prod_1 \times \prod_2$ of two polyhedra is a polyhedron. Furthermore, as a space, $\prod_1 \times \prod_2$ is the topological product of the spaces $\prod_1, \prod_2$.

7. Homology theory. The basic structure of a complex, particularly through No. 2, IV, provides the means for a wide application of algebraic methods. In discussing the resulting theory we shall follow the group-theoretic approach, which is basically due to E. Noether and which Pontrjagin's classical discoveries have made indispensable. We shall fully discuss finite complexes and give indications on the infinite case later.

Suppose, then, X finite. Take any additive topological group $G = \{g\}$ and form the expressions $C^p = g^i x_i^p$, $g^i \varepsilon G$, known as the *p-chains over G*. The set $\mathcal{C}^p(X, G) = \{C^p\}$ is an additive topological group, the direct product of α^p isomorphs of G, where α^p is the number of elements x_i^p. Introduce now a $(p-1)$-chain

$$F x_i^p = \sum_j [x_i^p : x_j^{p-1}] x_j^{p-1}$$

and define FC^p by linearity. This makes F a simultaneous homomorphism $\mathcal{C}^p(X, G) \to \mathcal{C}^{p-1}(X, G)$. The operator F is continuous and known as the *boundary operator*. It is a general operator defined for all complexes, very much as the differentiation operator d is defined for all differentiable functions. The chain FC^p is called the *boundary of* C^p, and $\mathcal{F}^p = \{FC^{p+1}\}$ is a subgroup of $\mathcal{C}^p(X, G)$, whose elements are called the *bounding p-chains over G*.

The p-chains mapped by F into zero (i.e. whose boundary is zero) are the *p-cycles over G.* Topologically speaking, G and hence $\mathcal{C}^p(X, G)$ are Hausdorff spaces. Therefore the p-cycles make up a closed subgroup $\mathcal{Z}^p(X, G)$ of $\mathcal{C}^p(X, G)$, the *group of the p-cycles over G. The basic axiom IV is immediately seen to be equivalent to the operator relation*

$$FF = 0,$$

or else to: *every boundary is a cycle*, or, again, to: *$\mathcal{F}^p$ is a subgroup of $\mathcal{Z}^p$.*

If G were discrete and likewise the groups $\mathcal{C}$, $\mathcal{Z}$, $\mathcal{F}$ we would now follow Poincaré and take the factor group $\mathcal{Z}/\mathcal{F}$. There are, however, strong reasons for requiring of topological groups that they be Hausdorff spaces; unfortunately, if, say, G, H are such groups with H a subgroup of G, the factor group G/H may fail to be a Hausdorff space when H is not closed in G. For this reason we take the closure $\overline{\mathcal{F}}^p$ (in $\mathcal{C}^p$), and since $\mathcal{F}^p \subset \mathcal{Z}^p$ and the latter is closed, we also have $\overline{\mathcal{F}}^p \subset \mathcal{Z}^p$. One may now safely introduce the factor group

$$\mathcal{H}^p(X, G) = \mathcal{Z}^p(X, G)/\overline{\mathcal{F}}^p(X, G),$$

known as the pth *homology group over G.* The importance of the homology groups will become clear as we proceed. At the present time it is sufficient to mention that, *when X is a finite geometric complex and is transformed topologically into a similar complex X_1, the two have the same homology groups.* That is to say, the homology groups are topological invariants. Thus a purely "algebraic" scheme leads to the discovery of such invariants. As we shall indicate at the end (No. 19, I), this scheme is so far-reaching that it may be applied even to general topological spaces. Notice that X, X_1 above need not look at all alike. Thus the surfaces of the cube and of the octahedron are very distinct complexes (polyhedra), but being topologically equivalent they have the same homology groups.

If the cycles γ^p, γ'^p are in the same homology class Γ^p (the same element of $\mathcal{H}^p(X, G)$), we express the fact, with Poincaré, by a *homology*, $\gamma^p \sim \gamma'^p$, and say that γ^p, γ'^p are *homologous.*

When X is infinite one may always consider finite chains, but their groups are to be taken as discrete throughout. Infinite chains over topological groups may be freely introduced only when X is star-finite. The groups $\mathcal{C}$, $\mathcal{Z}$, $\mathcal{F}$, $\mathcal{H}$ are then defined as before.

To illustrate the concepts introduced in the light of more familiar notions, take a geometric complex $K = \{\zeta\}$. A one-chain $g^i\zeta_i^1$ may be thought of as a path made up of "weighted" arcs (the ζ_i^1); a one-cycle corresponds rather nicely to the closed paths; a bounding one-cycle, to the paths which may be shrunk to points by a special kind of deformation in which one allows for canceling adjacent but oppositely oriented paths.

As a second example take a triangulated smooth closed surface $\sum$. All the two-cycles are simply multiples of the sum of the triangles suitably oriented, that is to say, in a certain sense multiples of $\sum$ itself. Again we see "cycle" associated with the absence of boundary. If $\sum$ were in $\mathcal{E}^3$ it could be shrunk to a point, and its cycles would in fact all be bounding. Thus "bounding" may suggest "shrinking down to points."

8. Coefficient groups. We have admitted fully general coefficient groups. In point of fact Poincaré considered only G = the group $\mathfrak{J}$ of the integers, giving rise to integral chains, etc. Tietze introduced the group $\mathfrak{J}_2$ of the residues mod 2; J. W. Alexander, the group $\mathfrak{J}_m$ of the residues mod m; Lefschetz, the rational group $\mathcal{R}$; and, finally, Pontrjagin, the general topological group G. The most important among the last is the group $\mathcal{P}$ of the real numbers mod 1. Among the more special categories a noteworthy one consists of Steenrod's *division-closure groups.* They are characterized by the following property: The subgroup $G(m)$ of the elements of the form mg is closed for every m. The class includes the discrete and the compact groups, and hence the groups $\mathfrak{J}_m$

(which are finite and therefore compact). For such groups and X finite $\bar{\mathcal{J}}^p = \mathcal{J}^p$, and so $\mathcal{H}^p = \mathcal{Z}^p/\mathcal{J}^p$.

Universal groups. Suppose that we consider groups $\mathcal{H}^p$ of a certain type. If G_0 is such that, when the groups $\mathcal{H}^p$ over G_0 are known, so are those over every G, then G_0 is said to be a *universal group* for the type in question. It is known that both $\mathfrak{J}$ and $\mathcal{P}$ are universal for a finite X (Pontrjagin), that $\mathfrak{J}$ is universal for a closure-finite X and finite cycles (Čech), and $\mathcal{P}$ for a star-finite X and infinite cycles (Steenrod).

Let us return to the situation envisioned by Poincaré: X finite and $G = \mathfrak{J}$ the group of the integers. The study of the groups $\mathcal{C}$, $\mathcal{Z}$, $\mathcal{J}$, $\mathcal{H}$ reduces, then, to a well-known problem on groups with a finite number of generators, and it is found that

$$\mathcal{H}^p = \mathcal{B}^p \times \mathcal{T}_1^p \times \cdots \times \mathcal{T}_s^p,$$

where $\mathcal{B}^p$ is a free group on, say, R^p generators and $\mathcal{T}_i^p$ a cyclic group of finite order t_i^p such that t_{i-1}^p divides t_i^p. The number R^p is the pth *Betti number* of X, and the t_i^p are its pth *torsion coefficients*.

On group-theoretical grounds a reasonable extension of the Betti numbers is this: Suppose $G = J$ a field; then $\mathcal{H}^p(X, J)$ is a vector space over J, and its dimension is by definition the pth *Betti number over* J, written $R^p(X, J)$. In practically all cases where such a number may be at all defined we have recently shown that it is the same for all groups of equal characteristic π (zero or a prime), and so it is the same for J and for $\mathfrak{J}_\pi$. For this reason the Betti numbers may as well be written $R^p(X, \pi)$. For the characteristic $\pi = 0$ the Betti number is the same as for the rational field, and it is in fact the number $R^p(X)$ already defined for the group of the integers.

For a geometric complex $\mathcal{K}$ the numbers $R^0(\mathcal{K}, \pi)$ are all equal, and their common value R^0 is the number of connected pieces of $\mathcal{K}$. This is the source of the early designations, connectivity indices or connectivities, given to the Betti numbers.

Returning to the general finite complex X we shall again let α^p denote the number of $x_i^p \varepsilon X$. We then have the *Euler-Poincaré relation*

$$(1) \qquad \chi(X) = \sum (-1)^p \alpha^p = \sum (-1)^p R^p(X, \pi).$$

The expression $\chi(X)$, which is thus independent of the characteristic π, is known as the *Euler-Poincaré characteristic of* X. It is very convenient for calculating the Betti numbers for low dimensions. Thus for the surface of genus p, $\sum_p$ we find $R^0 = 1$, $R^2 = 1$. An actual count shows that $\chi(\sum_p) = 2 - 2p$, and so $R^1 = 2p$.

9. Absolute and relative theory. Let X_1 be a closed subcomplex of X and $X_0 = X - X_1$ its open complement. We call (X_0, X_1) a *dissection* of X. As we have seen, both X_0, X_1 are complexes under the same assignment of $<$, dimension, $[\quad:\quad]$, as in X. Let us designate (temporarily) by F_i the boundary operator of X_i. It is readily found that $F_1 = F$ (F_1 takes the same values as F on X_1), while $F_0 = F \bmod F_1$ (i.e. neglecting elements in F_1). As a consequence, if C^p is a cycle of X_1 it is also a cycle of X, while on the contrary *a cycle of* X_0 *need not be a cycle of* X *itself*. For this reason the cycles of X_0 are termed *relative cycles* or, more precisely, *cycles of* X mod X_1. By contrast the cycles of X itself are called *absolute cycles*. The terms "absolute" and "relative" always refer to some complex X, which at the moment is the "universe of discourse" for all chains. Appropriate designations for the relative groups are: $\mathfrak{Z}^p(X, X_1)$, $\mathfrak{Z}^p(X, X_1, G)$, $\cdots$. The Euler-Poincaré relation (1) subsists with α^p the number of $x_i \varepsilon X_0$, and with the numbers $R^p(X, X_1, \pi)$ at the right.

It is worthy of note that the "relativization" which obtains here is well in line with the familiar relativization of topology. Thus let K be a geometric complex with the dissection (L_0, L_1). The number $R^0(K, L_1)$ is, this time, the number of connected pieces which do not meet L_1. A cycle of L_0 (cycle mod L_1) is a chain