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Singularities and Complex Geometry

Qi-keng Lu, Stephen S.-T. Yau, and
Anatoly Libgober, Editors

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Singularities and Complex Geometry

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and Complex Geometry
June 15-20, 1994
Beijing, China

Qi-keng Lu, Stephen S.-T. Yau,
and Anatoly Libgober, Editors

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PREFACE

The Seminar on Singularities and Complex Geometry was held at the Institute of Mathematics of the Chinese Academy, Beijing, China, from June 15 to 21, 1994. There were nearly 50 participants from China, USA, and Japan. This was a joint project of NSF of China and NSF of USA. Moreover, the seminar was also supported by the Bureau of International Cooperation, the Institute of Mathematics and the Institute of Applied Mathematics of the Chinese Academy of Sciences. We like to express thanks to all of these institutions.

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EMBEDDABLE CR-STRUCTURES AND DEFORMATIONS OF PSEUDOCONVEX SURFACES

JOHN BLAND+ AND C. L. EPSTEIN*

ABSTRACT. This is a somewhat expanded version of the lecture presented by the second author in Beijing, China at the International Conference on Complex Analysis and Singularities, June 19, 1994. Let V be a strongly pseudoconvex surface in $\mathbb{C}^N$ with smooth boundary ∂V and only normal interior singularities, for example the link of an isolated normal surface singularity. In this lecture we describe a *formal deformation theory* for the CR manifold ∂V and show that it is naturally isomorphic to the formal deformation theory for V. Among other things, we obtain a cohomology group defined on the ∂V which represents the first order moduli of the singularities of V. We also obtain a normal form for the formal deformations of the CR-structure and a non-linear system of equations whose formal solutions are precisely the aforementioned formal deformations. This system of equations is therefore a good candidate for the 2-dimensional analogue of Kuranishi's equations for the base space of a versal deformation of an isolated singularity in 3 or more dimensions. These results are proved in detail in [4].

1. INTRODUCTION

A complex manifold is a smooth even dimensional manifold V with a splitting of the complexified tangent space:

$$TV \otimes \mathbb{C} = T^{0,1}V \oplus T^{1,0}V, \quad T^{1,0}V = \overline{T^{0,1}V}.$$

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The subbundle $T^{0,1}V$ must satisfy the following conditions

1. fiber-$\dim_{\mathbb{C}} T^{0,1}V = \dim_{\mathbb{C}} V$,
2. $T^{1,0}V \cap T^{0,1}V =$ the zero section,
3. If $U \subset V$ is an open set and $Z, W \in \mathcal{C}^\infty(U, T^{0,1}V)$ then so does $[Z, W]$.

The third condition is called the integrability condition and it becomes very restrictive as soon as the $\dim_{\mathbb{C}} V > 1$. The Newlander-Nirenberg theorem asserts that to such an integrable complex structure is associated a coordinate atlas $\{(U_\alpha, \phi_\alpha)\}$ such that

$$\phi_{\alpha *} T^{0,1}V \mid_{U_\alpha} = T^{0,1}\mathbb{C}^n \mid_{\phi_\alpha(U_\alpha)} .$$

This implies that:

$$\partial_{\bar{z}_i} \phi_\alpha \circ \phi_\beta^{-1} = 0 \text{ on } \phi_\beta(U_\alpha \cap U_\beta), \quad i = 1, \ldots, n.$$

If $M \hookrightarrow V$ is a real hypersurface then the complex structure on V induces a structure on the complexified tangent space of M :

$$T^{0,1}M = TM \otimes \mathbb{C} \cap T^{0,1}V \mid_M .$$

This is called a CR-structure and it satisfies conditions analogous to those above:

1'. fiber-$\dim_{\mathbb{C}} T^{0,1}M = \frac{1}{2}(\dim_{\mathbb{R}} M - 1)$,
2'. $T^{1,0}M \cap T^{0,1}M =$ the zero section,
3'. If $U \subset M$ is an open set and $Z, W \in \mathcal{C}^\infty(U, T^{0,1}M)$ then so does $[Z, W]$.

In this case (3') is a non-trivial condition as soon as $\dim_{\mathbb{R}} M > 3$.

Besides their dimension, complex manifolds have no local invariants. This is not the case for CR-manifolds. If $\{Z_1, \ldots, Z_{n-1}\}$ is a local frame field for $T^{1,0}M$ then we can select a real vector field T so that $\{Z_1, \ldots, Z_{n-1}, \bar{Z}_1, \ldots, \bar{Z}_{n-1}, T\}$ defines an local framing for the full complexified tangent space with the induced orientation. The following Lie brackets computed modulo $T^{1,0}M \oplus T^{0,1}M$ define a matrix c_{ij} :

$$[Z_i, \bar{Z}_j] = i c_{ij} T \text{mod}\{Z_1, \ldots, Z_{n-1}, \bar{Z}_1, \ldots, \bar{Z}_{n-1}\}.$$

The Levi form is the Hermitian form defined on $T^{1,0}M$ by

$$\mathcal{L}(Z_i, Z_j) = c_{ij}.$$

The signature of the Levi form is defined independently of the choices involved in this definition. The CR-structure is strongly pseudoconvex if the Levi form is positive definite. The Levi form at $p \in M$ measures the order of contact between M at p and complex lines in

V. If M is strongly pseudoconvex at p then a complex line through p can be at most tangent to M at p and remains locally on one side of M.

A function, f defined on V is called holomorphic if

$$\bar{Z}f = 0 \text{ for all sections } \bar{Z} \text{ of } T^{0,1}V.$$

This is described invariantly by introducing the $\bar{\partial}$-operator, it is defined by

$$\bar{\partial}f = df \mid_{T^{0,1}V} .$$

The function, f is holomorphic if and only if $\bar{\partial}f = 0$. The restriction of a holomorphic function, f to a hypersurface M satisfies an analogous equation if $\dim_{\mathbb{R}} M \geq 3$:

$$\bar{Z}f \mid_M = 0 \text{ for all sections } \bar{Z} \text{ of } T^{0,1}M.$$

The invariant definition is in terms of the operator $\bar{\partial}_b$:

$$\bar{\partial}_b f = df \mid_{T^{0,1}M},$$

if the function f is the restriction of a holomorphic function then

$$\bar{\partial}_b f = 0.$$

A function that satisfies this equation is called a CR-function. If M is the strongly pseudoconvex boundary of compact complex manifold then actually this condition is also sufficient.

Proposition 1.1. [19] *If* $\dim_{\mathbb{R}} M \geq 3$ *and* M *is the strongly pseudoconvex boundary of compact complex manifold* V *then every function* f *defined on* M *which satisfies* $\bar{\partial}_b f = 0$ *has an extension to* V *as a holomorphic function.*

It is clear that under the hypotheses of this proposition questions about holomorphic functions on V can be rephrased in terms of questions about CR-functions on ∂V.

A CR-structure on an odd dimensional manifold M can also be defined intrinsically. It is simply a subbundle, $T^{0,1}M$ of the complexified tangent bundle with properties 1', 2', 3' as above. Given a compact manifold with a CR-structure the question of principal interest is whether or not it can be realized as the boundary of a compact complex space. If the structure is not strongly pseudoconvex then there are many examples that indicate that this is not usually the case. However if the dimension of M is at least five and the structure is strongly pseudoconvex then results of Rossi and Boutet de Monvel shows that this is always the case.

Theorem 1.2. [5], [16] *If M is a compact strongly pseudoconvex manifold with* $\dim_{\mathbb{R}} M > 3$ *then there is a normal Stein space V with smooth boundary such that*

$$\partial V = M.$$

Closely related to this problem is the question of embeddability. Let $(M, T^{0,1}M)$ denote a compact strongly pseudoconvex CR-manifold. We say that $(M, T^{0,1}M)$ is embeddable if there is a smooth embedding $\psi : M \longrightarrow \mathbb{C}^N$ such that:

$$\psi_* T^{0,1}M = T^{0,1}\psi(M). \tag{1.1}$$

This condition is equivalent to requiring the coordinate functions of ψ to be CR-functions. The right hand side of (1.1) denotes the CR-structure induced on a submanifold $N \hookrightarrow \mathbb{C}^N$ from the ambient complex structure on $\mathbb{C}^N$:

$$T^{0,1}N = TN \otimes \mathbb{C} \cap T^{0,1}\mathbb{C}^N.$$

This is a generalization of the construction defined above to the case of submanifolds of higher codimension. Using theorems of Harvey and Lawson, Boutet de Monvel and Kohn it follows that the property of embeddability is equivalent to the property that $(M, T^{0,1}M)$ arise as a compact hypersurface in a normal Stein space, see [9] and [10]. Indeed Boutet de Monvel proved

Theorem 1.3. [5] *If $(M, T^{0,1}M)$ is a compact strongly pseudoconvex CR-manifold with* $\dim_{\mathbb{R}} M > 3$ *then there is an CR-embedding of $(M, T^{0,1}M)$ into $\mathbb{C}^N$ for some N.*

In both the Rossi and Boutet de Monvel theorems three dimensional manifolds are excluded. In this case the fiber dimension of $T^{0,1}M$ is one and so the integrability condition, 3' is vacuous. This creates an abundance of CR-structures. Many of them are quite pathological in that they do not arise as the boundaries of compact complex spaces and consequently cannot be embedded into $\mathbb{C}^N$ for any N. The first example of this phenomenon in the literature is found in [16]. One begins with the induced structure on the unit three sphere. The fiber of $T^{0,1}\mathbb{S}^3$ is spanned at (z, w) by the vector field

$$\bar{Z}_0 = w\partial_{\bar{z}} - z\partial_{\bar{w}}.$$

If $\epsilon \neq 0$ is a small complex number then the CR-structure which is spanned at each point by the vector field

$$\bar{Z}_\epsilon = \bar{Z}_0 + \epsilon Z_0$$

cannot be realized as the boundary of any compact complex space. Indeed Burns showed the all solutions of the equation

$$\bar{Z}_\epsilon f = 0$$

are even functions if $\epsilon \neq 0$. Thus the CR-functions do not separate points on $\mathbb{S}^3$.

If $T^{0,1}M$ is a CR-structure on a compact three manifold then in a neighborhood, U of a given point there is a vector field $\bar{Z}$ such that

$$\mathcal{C}^\infty(U, T^{0,1}M) = f\bar{Z} \text{ where } f \in \mathcal{C}^\infty(U).$$

The local deformations of this structure are parametrized by functions $\varphi \in \mathcal{C}^\infty(U)$ with $\|\varphi\|_{L^\infty} < 1$. The deformed structure corresponding to φ has fiber at p spanned by $\bar{Z}_p + \varphi(p)Z_p$. A globally defined deformation is given by a section of the bundle $\mathrm{Hom}(T^{0,1}M, T^{1,0}M)$. If Φ is such a section then we denote the deformed structure by $T^{0,1}_\Phi M$, the fiber at p is given by

$$T^{0,1}_{\Phi p}M = \{\bar{Z} + \Phi_p(\bar{Z}) : \bar{Z} \in T^{0,1}_p M\}.$$

A result of Nirenberg refined by Jacobowits and Treves implies that, in the $\mathcal{C}^\infty$-topology on sections of $\mathrm{Hom}(T^{0,1}M, T^{1,0}M)$, the generic structure is not embeddable.

Associated to a CR-structure is a "$\bar{\partial}_b$-operator." The correspondence is as follows: a section Φ of $\mathrm{Hom}(T^{0,1}M, T^{1,0}M)$ is equivalent to a section of $\mathrm{Hom}((T^{1,0}M)^*, (T^{0,1}M)^*)$. We define $\bar{\partial}_b^\Phi$ by

$$\bar{\partial}_b^\Phi f = \bar{\partial}_b f + \Phi \circ \partial_b f.$$

Here the element $\partial_b f$ of $(T^{1,0}M)^*$ is defined by

$$\partial_b f = df \mid_{T^{1,0}M} .$$

A CR-function relative to the CR-structure $T^{0,1}_\Phi$ satisfies the equation

$$\bar{\partial}_b^\Phi f = 0.$$

It is clear that given such a differential operator one can reconstruct the subbundle $T^{0,1}_\Phi M$. In the sequel we use the notation $(M, \bar{\partial}_b)$ to denote the CR-manifold, M with the CR-structure defined by $\bar{\partial}_b$.

Over the last several years, along with several collaborators, we have been trying to understand when a CR-structure on a three dimensional manifold is embeddable. We take an embeddable structure $\bar{\partial}_b^0$ as a reference structure. Let V denote the variety in $\mathbb{C}^N$ bounded by $(M, \bar{\partial}_b^0)$. The deformations of $(M, \bar{\partial}_b^0)$ are naturally divided into three categories:

1. Structures which can be realized as deformations of the hypersurface within V. We call these wiggles.
2. Structures which can be realized as a hypersurface in a different Stein space.
3. Structures which are non-embeddable.

Ideally one would like to have a normal form for the CR-structures on M near to $T^{0,1}M$ in which each of these types is easily recognizable. Indeed the results in [12] and [3] lead to such a description for CR-structures on $\mathbb{S}^3$ near to the induced structure on the unit sphere.

In the Rossi example described above the deformed structures defined by $\bar{Z}_\epsilon$ are invariant under the $\mathbb{Z}_2$-action $(z, w) \longrightarrow (-z, -w)$. The reference structure, $\bar{Z}_0$ on $\mathbb{RP}^3 \simeq \mathbb{S}^3/\mathbb{Z}_2$ can be realized as an embedding via the map

$$\pm(z, w) \longrightarrow (z^2, \sqrt{2}zw, w^2).$$

The image of this map is

$$\{2xy - t^2 = 0\} \cap \{|(x, t, y)| = 1\}.$$

Even though $\bar{Z}_\epsilon$ is non-embeddable as a structure on $\mathbb{S}^3$ it is embeddable as a structure on $\mathbb{RP}^3$. Indeed there is a simple formula for the deformed embedding functions:

$$\pm(z, w) \longrightarrow (z^2 - \epsilon\bar{w}^2, \sqrt{2}(zw + \epsilon\bar{z}\bar{w}), w^2 - \epsilon\bar{z}^2).$$

The image lies in the variety

$$2xy - t^2 = -2\epsilon.$$

This family of CR-structures on $\mathbb{RP}^3$ corresponds to the versal deformation of the singularity at $(0, 0, 0)$ in the quadric $2xy - t^2 = 0$. These structures are therefore deformations of type 2 for the CR-manifold $(\mathbb{RP}^3, \bar{Z}_0)$. Note finally that $\varphi = \epsilon$ lies in the $\ker Z_0^2$.

In the lecture we describe results on the problem of identifying those CR-structures which correspond to deformations of the interior singularities in the normal Stein space V bounded by $(M, \bar{\partial}_b^0)$. It may indeed be the case that, modulo wiggles, these are the only deformations of type 2. It is in any case an essential problem to solve if one wants a reasonably complete description of the deformations of an embeddable CR-structure on a 3-manifold.

This is not a new problem. Indeed in [11], Kuranishi introduced a program for constructing the versal deformation of an isolated singularity by studying the integrable deformations of the CR-structure on a link of the singularity. He also introduced an additional condition to remove the effect of wiggles. His condition is simply that, to first order, the deformation should not be tangent to the initial variety. The construction of the base space of a versal deformation is reduced to the problem of constructing the solution space for a system of partial differential equations:

$$(K) \qquad \begin{aligned} I(\Phi) &= 0, \\ L(\Phi) &= 0. \end{aligned}$$

Here I is a non-linear operator given by the integrability condition, L is a linear operator that removes the wiggles. In dimensions 7 and above, Akahori, Miyajima, Buchweitz and Millson have shown that this approach does produce the versal deformation of an isolated singularity of depth 3. See for example [1], [2], [14], [15], [6] or the the expository paper of Millson, [13]. Millson's paper contains an extensive bibliography.

In dimension five there are technical difficulties which have stalled the analysis of (K). If the dimension is three, then this approach seems doomed to failure as the integrability condition is vacuous and Kuranishi's system degenerates. Motivated by the algebraic geometric definition of a formal deformation, we have introduced the notion of a formally embeddable perturbation of an embeddable CR-structure. If $(M, \bar{\partial}_b^0)$ denotes the reference CR-structure and V the normal Stein space it bounds, then under some mild technical hypotheses on V we have shown that the formal CR-deformation theory of $(M, \bar{\partial}_b^0)$ corresponds, to all orders to the formal deformation theory of V.

In [7] a non-linear map was introduced which takes a deformation of an embeddable CR-structure to a linear operator. Call this operator valued function $E(\Phi)$. One can show that the deformation defined by Φ is embeddable if and only if the rank of $E(\Phi)$ is finite. If $E(\Phi) = 0$ then Φ defines a "stably embeddable perturbation." This means that if X is **any** CR-embedding of $(M, \bar{\partial}_b^0)$ then $(M, \bar{\partial}_b^\Phi)$ can be realized as a small perturbation of $X(M)$. In this paper we show that the infinite order formal solutions of the equation

$$E(\sum_{i=1}^{\infty} t^i \Phi_i) \sim 0$$

are precisely the infinite order formally embeddable structures.

In addition we have defined a second order differential operator $\mathcal{P}$ whose range consists of first order deformations which are tangent to V. Introducing an L^2-structure on M to define adjoints we obtain a system of equations:

$$(K') \qquad \begin{aligned} E(\Phi) &= 0 \\ \mathcal{P}^*\Phi &= 0 \end{aligned}$$

which we think provides a reasonable three dimensional analogue of Kuranishi's system. As the system of equations, (K') does not seem to belong to any well understood class, its analysis will probably take a long time to complete.

The solution space to the linearization around $\Phi = 0$ of (K') has an interpretation as the *first order embeddable* deformations of $(M, \bar{\partial}_b^0)$. The dimension of this space is a numerical invariant of the CR-structure which we identify with the dimension of the space of first order deformations of V. For dimensions greater than three a similar result was proved by S.S.T. Yau, see [18].

In this lecture we briefly describe the notion of a formal deformation of a subvariety of $\mathbb{C}^N$ and the analogous concept of a formally embeddable CR-structure. We then state several results relating the two concepts. This leads to a normal form for formally embeddable CR-structures modulo a natural equivalence relation as well as the non-linear system (K'). Detailed proofs of these results can be found in [4].

2. Algebraic Deformation Theory

Suppose that V is an analytic surface with strongly pseudoconvex boundary in $\mathbb{C}^N$ and the components of $f = (f_1, \ldots, f_m)$ are generators for $\mathcal{I}_V$. We assume that these functions are analytic in an open set U containing V and globally generate $\mathcal{I}_V(U')$ for any open set U' with $\overline{U}' \subset\subset U$. In general these generators satisfy relations of the form

$$f \cdot p = \sum_{i=1}^{m} f_i p_i \equiv 0.$$

Here $p = (p_1, \ldots, p_m)$ is an m-tuple of functions analytic in a neighborhood of V. The germs of these relations define a coherent sheaf which we denote by $R(F)$.

Our discussion of deformation theory follows that presented in [17]. A deformation of V over a base T is an analytic space $\mathcal{V}$ together with a flat map

$$\pi : \mathcal{V} \longrightarrow T$$

and an isomorphism of analytic spaces

$$i : V \longrightarrow \pi^{-1}(\mathfrak{o}).$$