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CALCULUS OF VARIATIONS AND ITS APPLICATIONS

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EDITOR'S PREFACE

This volume contains the papers presented at the Eighth Symposium in Applied Mathematics, sponsored by the American Mathematical Society and the Office of Ordnance Research, and devoted to *The Calculus of Variations and Its Applications*. In addition to the nine invited addresses, there are included two brief notes, by P. G. Hodge, Jr., and by H. F. Weinberger, which were invited by the Program Committee and which embody discussion of the papers by D. C. Drucker and by J. L. Synge, respectively.

It seems obvious that one symposium could not profitably pay attention to all the directions in which variational methods have been applied. From the consultations of the Program Committee there resulted a group of addresses principally directed to applications in dynamics, but treating several other topics also.

The editor wishes to make special acknowledgment to the McGraw-Hill Book Company for their care in the production of the volume, and to all the authors for the careful preparation of their manuscripts. As a result the editor's task has been a comparatively light one.

LAWRENCE M. GRAVES
Editor

CONTENTS

ON VARIATIONAL PRINCIPLES IN ELASTICITY[1]

BY

ERIC REISSNER

1. Introduction. Boundary-value problems for the differential equations of the theory of elasticity have in common with many other differential-equation problems the property of being equivalent to problems of the calculus of variations. Recognition of this fact, for the problems of the elastic rod, goes back to Euler and Daniel Bernoulli. The general three-dimensional problem was first considered in this fashion by Green, in 1837.

We may, in the discussion of variational principles in elasticity, distinguish a number of phases as follows:

1. The formulation of different variational principles and their interrelation. The best-known examples of this are Green's minimum principle for displacements and Castigliano's maximum principle for stresses.

2. The application of variational principles to the establishment of approximate two- and one-dimensional theories for three-dimensional problems. A classical example of this is Kirchhoff's treatment of the differential equations and boundary conditions for transverse bending of thin plates.

3. The application of variational principles for the determination of numerical values of the solution of boundary-value problems.

4. The simultaneous use of different variational principles for the determination of upper and lower bounds of numerical values.

5. The use of variational principles for the proof of uniqueness and existence theorems in elasticity theory.

The present paper has as its object the consideration of some of the questions associated with phases 1 and 4, as they have been of interest to the author.

2. The boundary-value problem. We consider the following system of nine differential equations for six components of stress, $\tau_{ij} = \tau_{ji}$, and three components of displacement, u_i:

$$\tau_{ij,j} + \psi_{,u_i} = 0, \tag{1}$$

$$\tfrac{1}{2}(u_{i,j} + u_{j,i}) = W_{,\tau_{ij}}. \tag{2}$$

In these equations and in what follows we make use of the summation convention according to which one sums over repeated subscripts. A comma in front of a subscript denotes partial differentiation with respect to the variable in question, except that $f_{,i}$ indicates differentiation of f with respect to the Cartesian coordinate x_i.

[1] The work leading to this paper has been supported by the Office of Naval Research under Contract No. Nonr-1841(17) with the Massachusetts Institute of Technology.

The function ψ in the equilibrium equations is taken in the form

$$\psi = X_i u_i + \tfrac{1}{2} Y_{ij} u_i u_j, \tag{3}$$

where the X_i and $Y_{ij} = Y_{ji}$ are given functions of the coordinates x_i.

The function W in the stress-strain relations (2) is taken in the form

$$W = A_{ij}\tau_{ij} + \tfrac{1}{2} B_{ijkl}\tau_{ij}\tau_{kl}, \tag{4}$$

where the $A_{ij} = A_{ji}$ and $B_{ijkl} = B_{jikl} = B_{ijlk} = B_{jilk}$ are given functions of x_i.

The system (1) and (2) is to be solved in the interior of a region V with boundary surface S. We divide the surface S in two parts, S_u and S_p, and consider the following system of conditions:

$$\begin{aligned} &\text{On } S_u: \quad u_i = \phi_{,p_i}, \\ &\text{On } S_p: \quad p_i = \chi_{,u_i}. \end{aligned} \tag{5}$$

The functions ϕ and χ are taken in the form

$$\begin{aligned} \phi &= \bar{u}_i p_i + \tfrac{1}{2} b_{ij} p_i p_j, \\ \chi &= \bar{p}_i u_i + \tfrac{1}{2} c_{ij} u_i u_j, \end{aligned} \tag{6}$$

where $\bar{u}_i$, $\bar{p}_i$, $b_{ij} = b_{ji}$, $c_{ij} = c_{ji}$ are given functions of position on S_u and S_p, respectively. The quantities p_i are the x_i-components of the surface-stress intensity, given by

$$p_i = \cos(n, x_j)\,\tau_{ij}, \tag{7}$$

where n is the outward normal direction to the surface S.

The system of equations (1), (2), and (5), with ψ, W, ϕ, and χ defined by (3), (4), and (6), may be shown to represent the Euler equations and natural boundary conditions of a variational problem as stated below.

3. The general variational equation. Appropriate synthesis leads to the conclusion that a variational problem which has the differential equations (1) and (2) as Euler (differential) equations and the boundary conditions (5) as natural (or Euler) boundary conditions is the problem

$$\delta I = 0, \tag{8}$$

where

$$I = \int_V (\gamma_{ij}\tau_{ij} - \psi - W)\, dV - \int_{S_p} \chi\, dS - \int_{S_u} (p_i u_i - \phi)\, dS, \tag{9}$$

the quantities γ_{ij} being defined by

$$\gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{10}$$

and where the τ_{ij} and u_i are varied independently.[2]

[2] A variational theorem in which equations (10) are not used as definitions but are considered six of a total of fifteen differential equations for stresses, strains, and displacements has recently been formulated by K. Washizu in Technical Report 25-18 of the Aeroelastic and Structures Research Laboratory of the Massachusetts Institute of Technology (March, 1955).

To verify the correctness of the above statement, we write

$$(11)\quad \delta I = \int_V (\gamma_{ij}\,\delta\tau_{ij} + \tau_{ij}\,\delta\gamma_{ij} - \psi_{,u_i}\,\delta u_i - W_{,\tau_{ij}}\,\delta\tau_{ij})\,dV - \int_{S_p} \chi_{,u_i}\,\delta u_i\,dS - \int_{S_u} (u_i\,\delta p_i + p_i\,\delta u_i - \phi_{,p_i}\,\delta p_i)\,dS,$$

and we transform the second term in the volume integral by integration by parts, as follows:

$$(12)\quad \tfrac{1}{2}\int \tau_{ij}\,\delta(u_{i,j} + u_{j,i})\,dV = -\int \tau_{ij,j}\,\delta u_i\,dV + \int p_i\,\delta u_i\,dS.$$

Combination of (11) and (12) gives

$$(13)\quad \delta I = \int [(\gamma_{ij} - W_{,\tau_{ij}})\,\delta\tau_{ij} - (\tau_{ij,j} + \psi_{,u_i})\,\delta u_i]\,dV + \int_{S_p} (p_i - \chi_{,u_i})\,\delta u_i\,dS - \int_{S_u} (u_i - \phi_{,p_i})\,\delta p_i\,dS,$$

and this shows that the Euler equations of the problem are the differential equations (1) and (2) and the boundary conditions (5).

The variational theorem implied by (8) to (10) is a generalization of a theorem which was formulated earlier.[3] It reduces to the earlier theorem if it is assumed that the body force function ψ is absent and that the functions ϕ and χ in the boundary conditions are of the form

$$(6')\quad \phi = \bar{u}_i p_i, \qquad \chi = \bar{p}_i u_i.$$

What we have done in going from (6′) to (6) is to take the step from having either stress or displacement boundary conditions on S_p and S_u to a system of mixed boundary conditions on both S_p and S_u in such a manner as to preserve the form of the original theorem as a special case. Were it not for the desirability of accomplishing this within the framework of the generalized problem, there would, for the generalized problem, be no need for a separate consideration of the boundary portions S_p and S_u.

4. Variational equations for displacements or stresses. In order to bring out the significance of the general variational equation (8) for displacements *and* stresses, we state separately the less-general variational equations for displacements *or* stresses. In doing this, we are limiting ourselves here to stress and displacement boundary conditions of the form (6′).

a. Variational principle for displacements (*Green*). The stress-strain relations (2) are considered as equations of definition for the stresses (so that stress variations are dependent on displacement variations), and displacement variations are limited such that $\delta u_i = 0$ on S_u. Equations (2) are inverted and written, with the help of a function U, in the form

$$(14)\quad \tau_{ij} = U_{,\gamma_{ij}}.$$

[3] E. Reissner, *On a variational theorem in elasticity*, J. Math. Phys. vol. 29 (1950) pp. 90–95.

We further find that

(15) $$\gamma_{ij}\tau_{ij} - W = U,$$

and that the variational equation which has the equilibrium equations (1) and the stress boundary conditions in (6′) as Euler equations is of the form

(16) $$\delta I_u = 0,$$

where

(17) $$I_u = \int_V (U - \psi)\, dV - \int_{S_p} \bar{p}_i u_i\, dS.$$

b. Variational principle for stresses (Castigliano). We now assume that stress variations and displacement variations are such that all comparison states are equilibrium states. We then have $\delta p_i = 0$ on S_p and

(18) $$\delta(\tau_{ij,j} + \psi_{,u_i}) = 0$$

in the interior of the body.

We find that the variational equation which has the stress-strain relations (2) and the displacement boundary conditions in (6′) as Euler equations is of the form

(19) $$\delta I_\tau = 0,$$

where

(20) $$I_\tau = \int (-W - \psi + u_i\psi_{,u_i})\, dV + \int_{S_u} \bar{u}_i p_i\, dS.$$

We may note that, as long as ψ is a linear function of the u_i, which corresponds to the case of body forces independent of displacements, we have that $\psi - u_i\psi_{,u_i} = 0$ and therewith the disappearance of body-force terms in the variational equation. The extension of the principle to the case where ψ is a special quadratic function of the u_i, which allows use of the principle in connection with vibration problems, has been stated previously.[4]

5. A transformation and two inequalities. Useful information may be deduced from a comparison of the values of I for functions τ_{ij} and u_i which are not solutions of $\delta I = 0$ and for the functions τ_{ij} and u_i which are determined from $\delta I = 0$. We may designate the solution functions of $\delta I = 0$ by $\bar{\tau}_{ij}$ and $\bar{u}_i$ and write

(21) $$\tau_{ij} = \bar{\tau}_{ij} + \delta\tau_{ij}, \qquad u_i = \bar{u}_i + \delta u_i.$$

If we introduce (21) in (9), we shall have

(22) $$I = \int_V [(\bar{\gamma}_{ij} + \delta\gamma_{ij})(\bar{\tau}_{ij} + \delta\tau_{ij}) - \psi(\bar{u} + \delta u) - W(\bar{\tau} + \delta\tau)]\, dV$$
$$- \int_{S_p} \chi(\bar{u} + \delta u)\, dS - \int_{S_u} [(\bar{p}_i + \delta p_i)(\bar{u}_i + \delta u_i) - \phi(\bar{p} + \delta\bar{p})]\, dS.$$

[4] E. Reissner, *Note on the method of complementary energy*, J. Math. Phys. vol. 27 (1948) pp. 159–160.

We shall from now on in this section limit ourselves to the case for which ψ, χ, and ϕ are linear functions and W is a homogeneous second-degree function. We then have

$$\psi(\bar{u} + \delta u) = X_i\bar{u}_i + X_i\,\delta u_i, \tag{23a}$$
$$\chi(\bar{u} + \delta u) = \bar{p}_i\bar{u}_i + \bar{p}_i\,\delta u_i, \tag{23b}$$
$$\phi(\bar{p} + \delta p) = \bar{u}_i\bar{p}_i + \bar{u}_i\,\delta p_i, \tag{23c}$$

and

$$W(\bar{\tau} + \delta\tau) = W(\bar{\tau}) + W_{,\bar{\tau}_{ij}}\,\delta\tau_{ij} + W(\delta\tau). \tag{24}$$

We further write

$$I = \bar{I} + \delta I + \delta^2 I, \tag{25}$$

where $\bar{I}$ is the value of I when $\tau_{ij} = \bar{\tau}_{ij}$ and $u_i = \bar{u}_i$, where δI contains all terms linear in the variations $\delta\tau_{ij}$ and δu_i, and where $\delta^2 I$ contains all terms of second degree in the variations.

Rearrangement of terms in (22) gives us

$$\bar{I} = \int_V [\bar{\gamma}_{ij}\bar{\tau}_{ij} - X_i\bar{u}_i - W(\bar{\tau})]\,dV - \int_{S_p} \bar{p}_i\bar{u}_i\,dS - \int_{S_u} (\bar{u}_i - \bar{u}_i)\bar{p}_i\,dS, \tag{26}$$
$$\delta I = 0, \tag{27}$$

(as it should be), and

$$\delta^2 I = \int_V [\delta\gamma_{ij}\,\delta\tau_{ij} - W(\delta\tau)]\,dV - \int_{S_u} \delta p_i\,\delta u_i\,dS. \tag{28}$$

Equations (26) and (28) may be simplified if account is taken of some of the basic relations. Since W is homogeneous of the second degree, we have

$$W(\bar{\tau}) = \tfrac{1}{2}W_{,\bar{\tau}_{ij}}\bar{\tau}_{ij}. \tag{29}$$

Furthermore, $\bar{\gamma}_{ij} = W_{,\bar{\tau}_{ij}}$ while $\bar{u}_i = \bar{u}_i$ on S_u, and $\bar{p}_i = \bar{p}_i$ on S_p. Therewith

$$\bar{I} = \int_V (\tfrac{1}{2}\bar{\gamma}_{ij}\bar{\tau}_{ij} - X_i\bar{u}_i)\,dV - \int_{S_p} \bar{p}_i\bar{u}_i\,dS. \tag{30}$$

We further have

$$\int \bar{\gamma}_{ij}\bar{\tau}_{ij}\,dV = -\int \bar{\tau}_{ij,j}\bar{u}_i\,dV + \int \bar{p}_i\bar{u}_i\,dS, \tag{31}$$

and, since $\bar{\tau}_{ij,j} + X_i = 0$, finally

$$\bar{I} = -\tfrac{1}{2}\int_V X_i\bar{u}_i\,dV - \tfrac{1}{2}\int_{S_p} \bar{p}_i\bar{u}_i\,dS + \tfrac{1}{2}\int_{S_u} \bar{p}_i\bar{u}_i\,dS. \tag{32}$$

In order to transform $\delta^2 I$ as given by (28), we have at our disposal the relations

$$W(\delta\tau) = \tfrac{1}{2}W_{,\delta\tau_{ij}}\delta\tau_{ij} \tag{33}$$

and

$$\int \delta\gamma_{ij}\,\delta\tau_{ij}\,dV = -\int \delta\tau_{ij,j}\,\delta u_i\,dV + \int \delta p_i\,\delta u_i\,dS. \tag{34}$$

It is not immediately apparent in which way to utilize these two facts. However, let us write $\delta^2 I$ in the following two alternate forms:

$$\delta^2 I = \int [-\delta\tau_{ij,j}\,\delta u_i - W(\delta\tau)]\,dV + \int_{S_p} \delta p_i\,\delta u_i\,dS \tag{35}$$

or

$$\delta^2 I = \int [(\delta\gamma_{ij} - W_{,\delta\tau_{ij}})\,\delta\tau_{ij} + W(\delta\tau)]\,dV - \int_{S_u} \delta p_i\,\delta u_i\,dS. \tag{36}$$

In general, the quantity $\delta^2 I$ may be made both positive and negative by a suitable choice of the integrands. There are two exceptional cases where this is not so. These cases are given when

$$\delta\tau_{ij,j} = 0 \text{ in } V \quad \text{and} \quad \delta p_i = 0 \text{ on } S_p, \tag{37}$$

or when

$$\delta\gamma_{ij} - W_{,\delta\tau_{ij}} = 0 \text{ in } V \quad \text{and} \quad \delta u_i = 0 \text{ on } S_u. \tag{38}$$

We now take account of the fact that the function W is positive-definite. Accordingly, when (37) holds, we have $\delta^2 I \leqq 0$, and when (38) holds, we have $0 \leqq \delta^2 I$. We note that (37) represents the same limitations on variations as those associated with the variational principle for stresses [equation (19)] and that (38) represents the same limitations on variations as those associated with the variational principle for displacements [equation (16)]. We conclude then from (25) and (27) that the following basic inequality holds:

$$I_\tau \leq \bar{I} \leqq I_u. \tag{39}$$

Equation (39) confirms the known fact that, in the variational theorem for displacements, one is concerned with a minimum problem and, in the variational theorem for stresses, one is concerned with a maximum problem. In contrast to this, the general variational theorem for stresses and displacements is no more than a stationary-value problem.

What is of importance in the present demonstration of this known fact is the explicit way in which both the minimum principle for displacements and the maximum principle for stresses are seen to be direct consequences of a more general principle for stresses and displacements.

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VARIATIONAL PRINCIPLES IN THE MATHEMATICAL THEORY OF PLASTICITY[1]

BY

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1. Introduction. The fundamental definitions of work hardening and perfect plasticity have been shown to have strong implications with respect to uniqueness of solution for elastic-plastic bodies. It is not surprising, therefore, to find that they lead rather directly to the variational principles as well. Perfect-plasticity theory and both the incrementally linear and the incrementally nonlinear theories for work-hardening materials are considered. The several counterparts of the minimum-potential-energy and the minimum-complementary-energy theorems are derived in a unified manner for stress-strain relations of great generality. Absolute-minimum principles rather than relative ones are established.

There are any number of approaches to the establishment of variational principles. One is to state the principles directly and then proceed to prove them. Although clear and precise statements can be made, the motivation for the original inspiration does not appear. A newcomer to the field then frequently will be unable to appreciate the development and generally will not see how to produce appropriate theorems or modifications of his own. The approach to be followed here does not suppose the result to be known in advance. It is synthetic in a sense, because the basic theorems have been stated and proved for a number of special materials [1–6].[2] Nevertheless, it is a procedure which arises logically from fundamental postulates in elasticity and in plasticity theory, and it is systematic.

In the theory of elasticity, whether linear or nonlinear, the steps are reasonably straightforward. The equation of virtual work is written first under the implicit assumption of continuity of displacement and what may be termed equilibrium continuity of the stresses (surface tractions must be continuous across any surface, but the normal stress components parallel to the surface may be discontinuous). In a common notation, repeated subscripts indicating summation,

$$\int_A T_i^* u_i \, dA + \int_V F_i^* u_i \, dV = \int_V \sigma_{ij}^* \epsilon_{ij} \, dV. \tag{1}$$

The starred quantities are related through equilibrium, and the unstarred are compatible. There need be no relation between the two sets of quantities. For convenience, the surface area A is divided into the region A_T, on which the surface tractions T_i are specified, and the region A_u, over which displace-

[1] The results presented in this paper were obtained in the course of research conducted under Contract Nonr 562(10) between the Office of Naval Research and Brown University.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

ments u_i are given. The true and unique solution (no buckling, no initial stress) to the boundary-value problem with given body forces F_i thus satisfies

$$(2) \qquad \int_V \sigma_{ij}^t \epsilon_{ij}^t \, dV - \int_{A_u} T_i^t u_i \, dA - \int_{A_T} T_i u_i^t \, dA - \int_V F_i u_i^t \, dV = 0.$$

If approximate solutions are sought, two procedures suggest themselves immediately. One is to choose a compatible strain-displacement field ϵ_{ij}^c, u_i^c and satisfy the boundary conditions on A_u. The other is to select an equilibrium stress field σ_{ij}^E which satisfies the surface-traction boundary conditions on A_T. More elaborate mixed schemes may be devised, but they cannot be classed as obvious [3].

The value of an approximation procedure, or of a guess, must be determined by comparison of the approximate solution with the unknown true answer. The real difficulty and the intuitive heart of the problem lie in the decision on what should be compared. As has been noted, the equation of virtual work will be satisfied if the natural strains and displacements are replaced by any chosen set satisfying compatibility and the boundary conditions on A_u. Therefore

$$(3) \quad \int_V \sigma_{ij}^t \epsilon_{ij}^c \, dV - \int_{A_T} T_i u_i^c \, dA - \int_V F_i u_i^c \, dV = \int_V \sigma_{ij}^t \epsilon_{ij}^t \, dV - \int_{A_T} T_i u_i^t \, dA - \int_V F_i u_i^t \, dV.$$

Transposing and calling the difference between the true and the assumed solution $\Delta\epsilon_{ij}$, Δu_i,

$$(4) \qquad \int_V \sigma_{ij}^t \, \Delta\epsilon_{ij} \, dV - \int_{A_T} T_i \, \Delta u_i \, dA - \int_V F_i \, \Delta u_i \, dV = 0.$$

This form suggests strongly a consideration of the elastic-strain energy density written as a function of strain alone,

$$(5) \qquad W(\epsilon_{ij}) = \int_0^{\epsilon_{ij}} \sigma_{ij} \, d\epsilon_{ij},$$

because $dW = (\partial W / \partial \epsilon_{ij}) \, d\epsilon_{ij} = \sigma_{ij} \, d\epsilon_{ij}$. Equation (4) can then be restated as

$$(6) \qquad \delta_e \left[\int_V W(\epsilon_{ij}^t) \, dV - \int_{A_T} T_i u_i^t dA - \int_V F_i u_i^t \, dV \right] \equiv \delta[\text{P.E.}^t] = 0,$$

where δ_e is to be interpreted as the first variation of the expression in brackets (the potential energy) as u_i and ϵ_{ij} are varied in accordance with compatibility and the boundary conditions on u_i.

A variational principle is not necessarily very helpful in solving problems. The assumed state may not be close to the true one. What is required instead is an absolute-maximum or -minimum principle, a comparison of the value of the potential energy for the assumed state with that of the true state, without restriction on the magnitude of the difference between the states. The presentation here is, however, within the framework of small-displacement theory.

A comparison may be made with the aid of the identity

$$(7)\quad \int_V W(\epsilon_{ij}^c)\,dV - \int_{A_T} T_i u_i^c\,dA - \int_V F_i u_i^c\,dV = \int_V W(\epsilon_{ij}^t)\,dV - \int_{A_T} T_i u_i^t\,dA$$
$$- \int_V F_i u_i^t\,dV + \int_V [W(\epsilon_{ij}^c) - W(\epsilon_{ij}^t)]\,dV - \int_{A_T} T_i\,\Delta u_i\,dA - \int_V F_i\,\Delta u_i\,dV.$$

In view of (4), therefore, the potential energy of any admissible compatible state is algebraically more than the potential energy of the true state by

$$(8)\qquad \text{P.E.}^c - \text{P.E.}^t = \int_V [W(\epsilon_{ij}^c) - W(\epsilon_{ij}^t) - \sigma_{ij}^t\,\Delta\epsilon_{ij}]\,dV.$$

The integrand may be rewritten as

$$(9)\qquad \int_0^{\epsilon_{ij}^c} \sigma_{ij}\,d\epsilon_{ij} - \int_0^{\epsilon_{ij}^t} \sigma_{ij}\,d\epsilon_{ij} - \sigma_{ij}^t\,\Delta\epsilon_{ij} = \int_{\epsilon_{ij}^t}^{\epsilon_{ij}^c} (\sigma_{ij} - \sigma_{ij}^t)\,d\epsilon_{ij}.$$

The rectangles in Fig. 1 symbolize $\sigma_{ij}^t\,\Delta\epsilon_{ij}$. The shaded triangles represent (9), the integrand of (8). In Fig. 1*a* to *c* the triangles are on the positive-strain-energy side of the symbolic stress-strain curves for any magnitude $\Delta\epsilon_{ij}$.

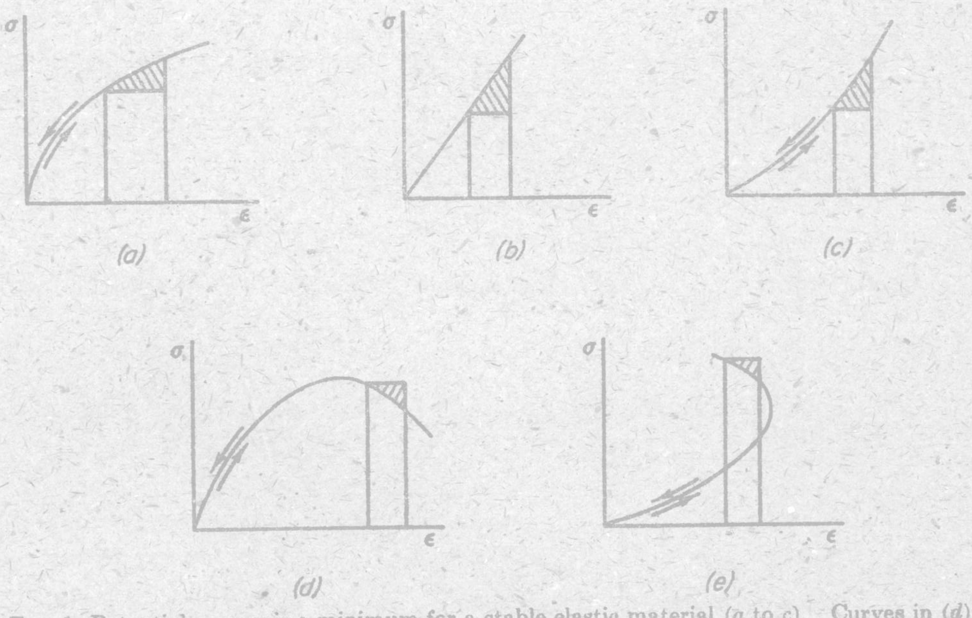

FIG. 1. Potential energy is a minimum for a stable elastic material (*a* to *c*). Curves in (*d*) and (*e*) are for an unstable material.

The potential energy is an absolute minimum for a linear or for a stable non-linear elastic material. For an unstable material (Fig. 1*d* and *e*), the shaded triangles are on the negative side for some $\Delta\epsilon_{ij}$, and the potential energy is not an absolute minimum.

A similar set of steps leads to the principle of minimum complementary energy. Equation (2) is satisfied if the σ_{ij}^t, T_i, F_i system is replaced by any other in equilibrium. For any state of stress σ_{ij}^E which satisfies the boundary conditions on A_T and is in equilibrium with F_i,

$$\int_V \Delta\sigma_{ij}\epsilon_{ij}^t \, dV - \int_{A_u} \Delta T_i u_i \, dA = 0, \tag{10}$$

where $\sigma_{ij}^E - \sigma_{ij}^t = \Delta\sigma_{ij}$ and ΔT_i is the corresponding change in surface traction on A_u.

The complementary-energy density as a function of stress alone is suggested by the first integral:

$$\Omega(\sigma_{ij}) = \int_0^{\sigma_{ij}} \epsilon_{ij} \, d\sigma_{ij}, \tag{11}$$

because $d\Omega = (\partial\Omega/\partial\sigma_{ij}) \, d\sigma_{ij} = \epsilon_{ij} \, d\sigma_{ij}$. Equation (10) can then be restated as

$$\delta_\sigma \left[\int_V \Omega(\sigma_{ij}^t) \, dV - \int_{A_u} T_i^t u_i \, dA \right] \equiv \delta_\sigma[\text{C.E.}^t] = 0, \tag{12}$$

where δ_σ is to be interpreted as the first variation of the expression in brackets (complementary energy) as σ_{ij} and T_i are varied in accordance with equilibrium

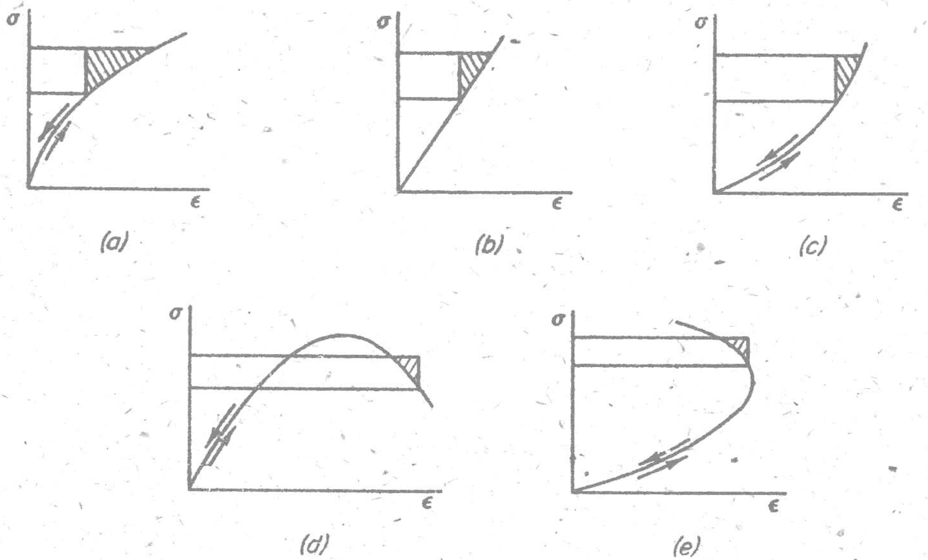

Fig. 2. Complementary energy is a minimum for a stable elastic material (a to c). Curves in (d) and (e) are for an unstable material.

with F_i and the boundary conditions on A_T. Figure 2 is symbolic of the fact that the complementary energy is an absolute minimum for a stable material. Corresponding to (8) and (9),

$$\text{C.E.}^E - \text{C.E.}^t = \int_V [\Omega(\sigma_{ij}^E) - \Omega(\sigma_{ij}^t) - \epsilon_{ij}^t \, \Delta\sigma_{ij}] \, dV, \tag{13}$$

where the integrand may be rewritten as

$$\int_{\sigma_{ij}^t}^{\sigma_{ij}^E} (\epsilon_{ij} - \epsilon_{ij}^t) \, d\sigma_{ij}. \tag{14}$$

The symbolic representation of the stress and strain tensors by one dimension each in Figs. 1 and 2 and the terms stable and unstable can be given general meaning and made precise.

2. The fundamental postulate for elasticity and plasticity. A basic postulate has been formulated for both elastic and plastic media [4] without time effects. It is essentially a definition of a stable material and may be stated as follows:

No work can be extracted from the material and the system of forces acting upon it.

A more useful statement is in terms of an external agency which applies a set of additional forces to the body under a given load and then removes the added forces. The external agency must do positive work in the application of force. Over the cycle of application and removal the work done *by the external agency* must be positive if plastic deformation occurs in work-hardening material and will be zero if only elastic changes take place. For a perfectly plastic material, the work done by the external agency may also be zero when plastic deformation takes place, although generally it will be positive.

The basic postulate may be applied to a homogeneous material under homogeneous stress σ_{ij}^a and strain ϵ_{ij}^a. Suppose that the external agency changes the state of stress by $\Delta\sigma_{ij}$ to σ_{ij}^b. The strain will change by $\Delta\epsilon_{ij}$ to ϵ_{ij}^b. Then the postulate requires

$$\int_{\epsilon_{ij}^a}^{\epsilon_{ij}^b} (\sigma_{ij} - \sigma_{ij}^a)\, d\epsilon_{ij} > 0. \tag{15}$$

The value of the integral is strongly path-dependent in the plastic range but is, of course, independent of path for any elastic material.

3. Absolute-minimum principles in elasticity. The inequality (15) is a formal expression of the requirement that the shaded triangles of Fig. 1 be on the positive-strain-energy side of the stress-strain curve. Although for a nonlinear elastic material ϵ_{ij}^b depends upon ϵ_{ij}^a as well as upon $\Delta\sigma_{ij}$, the integral is path-independent. Choosing a straight-line path in stress space from σ_{ij}^a to σ_{ij}^b, it is obvious that inequality (15) may be continued as

$$0 < \int_{\epsilon_{ij}^a}^{\epsilon_{ij}^b} (\sigma_{ij} - \sigma_{ij}^a)\, d\epsilon_{ij} < (\sigma_{ij}^b - \sigma_{ij}^a)(\epsilon_{ij}^b - \epsilon_{ij}^a) \equiv \Delta\sigma_{ij}\,\Delta\epsilon_{ij}. \tag{16}$$

Also, from

$$\begin{aligned}(\sigma_{ij}^b - \sigma_{ij}^a)(\epsilon_{ij}^b - \epsilon_{ij}^a) &= \int_a^b d[(\sigma_{ij} - \sigma_{ij}^a)(\epsilon_{ij} - \epsilon_{ij}^a)] \\ &= \int_{\epsilon_{ij}^a}^{\epsilon_{ij}^b} (\sigma_{ij} - \sigma_{ij}^a)\, d\epsilon_{ij} + \int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\epsilon_{ij} - \epsilon_{ij}^a)\, d\sigma_{ij},\end{aligned} \tag{17}$$

$$0 < \int_{\sigma_{ij}^a}^{\sigma_{ij}^b} (\epsilon_{ij} - \epsilon_{ij}^a)\, d\sigma_{ij} < \Delta\sigma_{ij}\,\Delta\epsilon_{ij}. \tag{18}$$

Inequality (18) expresses the requirement that the shaded triangles be as shown in Fig. 2*a* to *c* and not as in Fig. 2*d* and *e*.

Materials of the type of Figs. 1*d* and *e* and 2*d* and *e* are thus excluded from our consideration, although not necessarily from physical reality. For elastic materials which follow the postulated behavior, comparison of (16) with (9) and of (18) with (14) proves that the potential energy and the complementary energy of the true state are both an absolute minimum:

$$\text{(19)} \qquad \begin{aligned} \text{P.E.}^t &\leqq \text{P.E.}^c, \\ \text{C.E.}^t &\leqq \text{C.E.}^E, \end{aligned}$$

the equality sign applying only to the trivial case of the admissible state c or E coinciding with the true state.

4. Deformation or total theories of plasticity. If no distinction is made between loading and unloading, or if each point of the body is assumed to be at the maximum load intensity in its history, deformation theories postulate a unique relation between stress and total strain. Although physically unacceptable, in general, because plastic deformation is path-dependent and irreversible, such theories do in some instances lead to very useful results. Under the assumptions mentioned, there is no need to consider deformation theory further, as the theory is indistinguishable from nonlinear elasticity. No matter how elaborate the stress-strain relation, if the material postulated is stable, the principles of minimum potential and minimum complementary energy apply without any change.

If, on the other hand, loading is taken to be nonlinear whereas unloading is assumed to follow a linear elastic relation, the inconsistency of deformation theory becomes of primary importance. The mathematical and physical meaning of solutions then becomes quite obscure.

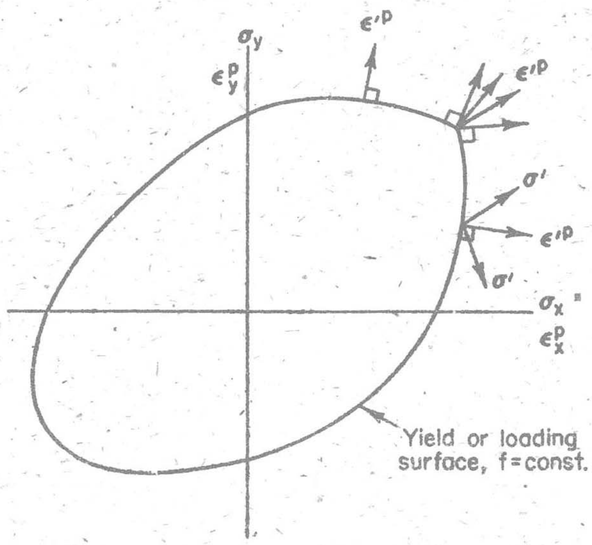

FIG. 3. Normality of plastic-strain increment (rate).