Probability and
Its Applications

K. L. Chung
R. J. Williams

Introduction to Stochastic Integration

Second Edition

随机积分导论 第2版

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Cover Image. A Brownian motion B starts at a point x inside the domain D and first leaves D at B_τ. Under conditions explained in Section 6.4, a solution to the Schrödinger equation $\frac{1}{2}\Delta\psi + q\psi = 0$ in D that approaches f on the boundary of D can be represented by

$$\psi(x) = E^x\left[\exp\left(\int_0^\tau q(B_t)\,dt\right) f(B_\tau)\right].$$

The Brownian path used in this illustration is from The Fractal Geometry of Nature © 1982 by Benoit B. Mandelbrot and is used with his kind permission.

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PREFACE

This is a substantial expansion of the first edition. The last chapter on stochastic differential equations is entirely new, as is the longish section §9.4 on the Cameron-Martin-Girsanov formula. Illustrative examples in Chapter 10 include the warhorses attached to the names of L. S. Ornstein, Uhlenbeck and Bessel, but also a novelty named after Black and Scholes. The Feynman-Kac-Schrödinger development (§6.4) and the material on reflected Brownian motions (§8.5) have been updated. Needless to say, there are scattered over the text minor improvements and corrections to the first edition. A Russian translation of the latter, without changes, appeared in 1987.

Stochastic integration has grown in both theoretical and applicable importance in the last decade, to the extent that this new tool is now sometimes employed without heed to its rigorous requirements. This is no more surprising than the way mathematical analysis was used historically. We hope this modest introduction to the theory and application of this new field may serve as a text at the beginning graduate level, much as certain standard texts in analysis do for the deterministic counterpart. No monograph is worthy of the name of a true textbook without exercises. We have compiled a collection of these, culled from our experiences in teaching such a course at Stanford University and the University of California at San Diego, respectively. We should like to hear from readers who can supply more and better exercises.

A word about the exposition. We have consistently chosen clarity over brevity. As one of the authors suggested elsewhere, readers who insist on concision are free to skip every other line or so. But be warned that most errors in mathematics are concealed under the surreptitious cover of terseness, whereas a fuller exposure leaves less to pitfalls. A good example of our preference is afforded by the demonstration in §10.3 of the Markov property for the family of solutions of a stochastic differential equation, which is often glossed over in texts and *ergo* gloated over by innocent readers. Actually, the point at issue there is subtle enough to merit the inculcation.

For the new material, the following acknowledgements are in order. Michael Sharpe provided helpful comments on several points in Chapter 2. Giorgio Letta inspired an extension of predictable integrability in Chapter 3. Martin Barlow supplied two examples in Chapter 3. Daniel Revuz and Marc Yor permitted the references to their forthcoming book. Darrell Duffie gave lectures on the Black-Scholes model in Chung's class during 1986 which led to its inclusion in §10.5. We wish also to thank those colleagues and students who contributed comments and corrections to the first edition. Lisa Taylor helped with the proof reading of this edition. Kathleen Flynn typed the manuscipt of the first edition in TEX, whilst artists at Stanford Word Graphics, especially Walter Terluin, added final touches to the figures. We are appreciative of the interest and cooperation of the staff at Birkhäuser Boston. Indeed, the viability of this new edition was only an optional, not a predictable event when we prepared its precursor in 1983.

March 1990

K. L. Chung
R. J. Williams

PREFACE TO THE FIRST EDITION

The contents of this monograph approximate the lectures I gave in a graduate course at Stanford University in the first half of 1981. But the material has been thoroughly reorganized and rewritten. The purpose is to present a modern version of the theory of stochastic integration, comprising but going beyond the classical theory, yet stopping short of the latest discontinuous (and to some distracting) ramifications. Roundly speaking, integration with respect to a local martingale with continuous paths is the primary object of study here. We have decided to include some results requiring only right continuity of paths, in order to illustrate the general methodology. But it is possible for the reader to skip these extensions without feeling lost in a wilderness of generalities. Basic probability theory inclusive of martingales is reviewed in Chapter 1. A suitably prepared reader should begin with Chapter 2 and consult Chapter 1 only when needed. Occasionally theorems are stated without proof but the treatment is aimed at self-containment modulo the inevitable prerequisites. With considerable regret I have decided to omit a discussion of stochastic differential equations. Instead, some other applications of the stochastic calculus are given; in particular Brownian local time is treated in detail to fill an unapparent gap in the literature. The applications to storage theory discussed in Section 8.4 are based on lectures given by J. Michael Harrison in my class. The material in Section 8.5 is Ruth Williams's work, which has now culminated in her dissertation [77].

At the start of my original lectures, I made use of Métivier's lecture notes [59] for their ready access. Later on I also made use of unpublished notes on continuous stochastic integrals by Michael J. Sharpe, and on local time by John B. Walsh. To these authors we wish to record our indebtedness. Some oversights in the references have been painstakingly corrected here. We hope any oversight committed in this book will receive similar treatment.

A methodical style, due mainly to Ruth Williams, is evident here. It is not always easy to strike a balance between utter precision and relative readability, and the final text represents a compromise of sorts. As a good author once told me, one cannot really hope to achieve consistency in writing a mathematical book—even a small book like this one.

December 1982 K. L. Chung

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$\int_{(s,t]} X\,dM$	39
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a.e. almost everywhere	
a.s. almost surely	
c.	57
r.c.	57
l.c.	57
r.c.l.l.	57
l.c.r.l.	57
i.o. infinitely often	
▮ end of proof	

1

PRELIMINARIES

1.1 Notations and Conventions

For each interval I in $\mathbb{R} = (-\infty, \infty)$ let $\mathcal{B}(I)$ denote the σ-field of Borel subsets of I. For each $t \in \mathbb{R}_+ = [0, \infty)$, let $\mathcal{B}_t$ denote $\mathcal{B}([0,t])$ and let $\mathcal{B}$ denote $\mathcal{B}(\mathbb{R}_+) = \bigvee_{t \in \mathbb{R}_+} \mathcal{B}_t$ — the smallest σ-field containing $\mathcal{B}_t$ for all t in $\mathbb{R}_+$. Let $\overline{\mathbb{R}}_+ = [0, \infty]$ and $\overline{\mathcal{B}}$ denote the Borel σ-field of $\overline{\mathbb{R}}_+$ generated by $\mathcal{B}$ and the singleton $\{\infty\}$. Let λ denote the Lebesgue measure on $\mathbb{R}$.

Whenever t appears without qualification it denotes a generic element of $\mathbb{R}_+$. The collection $\{x_t, t \in \mathbb{R}_+\}$ is frequently denoted by $\{x_t\}$. The parameter t is sometimes referred to as time.

Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{N}_0$ denote $\mathbb{N} \cup \{0\}$, and $\mathbb{N}_\infty$ denote $\mathbb{N} \cup \{\infty\}$. Whenever n, k, or m, appears without qualification, it denotes a generic element of $\mathbb{N}$. A sequence $\{x_n, n \in \mathbb{N}\}$ is frequently denoted by $\{x_n\}$. We write $x_n \to x$ when $\{x_n\}$ converges to x. A sequence of real numbers $\{x_n\}$ is said to be increasing (decreasing) if $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$) for all n. The notation $x_n \uparrow x$ ($x_n \downarrow x$) means $\{x_n\}$ is increasing (decreasing) with limit x.

For each $d \in \mathbb{N}$, the components of $x \in \mathbb{R}^d$ are denoted by x_i, or sometimes by x^i, $1 \le i \le d$, and the Euclidean norm of x is denoted by $|x| = \left(\sum_{i=1}^{d}(x_i)^2\right)^{\frac{1}{2}}$.

The symbol 1_A denotes the indicator function of a set A, i.e., $1_A(x) = 1$ if $x \in A$ and $= 0$ if $x \notin A$. The symbol $\emptyset$ denotes the empty set.

For each n, $C^n(\mathbb{R})$ or simply C^n denotes the set of all real-valued continuous functions defined on $\mathbb{R}$ for which the first n derivatives exist and are continuous. We use $C(\mathbb{R})$ to denote the set of real-valued continuous functions on $\mathbb{R}$ and $C^\infty(\mathbb{R})$ or C^∞ to denote $\bigcap_{n\in\mathbb{N}} C^n$, the set of infinitely differentiable real-valued functions on $\mathbb{R}$.

We use the words "positive", "negative", "increasing", and "decreasing", in the loose sense. For example, "x is positive" means "$x \ge 0$"; the qualifier "strictly" is added when "$x > 0$" is meant. The infimum of an empty set of real numbers is defined to be ∞. A sum over an empty index set is defined to be zero.

1.2 Measurability, L^p Spaces and Monotone Class Theorems

Suppose (S, Σ) is a measurable space, consisting of a non-empty set S and a σ-field Σ of subsets of S. A function $X : S \to \mathbb{R}^d$ is called Σ-measurable if $X^{-1}(A) \in \Sigma$ for all Borel sets A in $\mathbb{R}^d$, where X^{-1} denotes the inverse image. A similar definition holds for a function $X : S \to \overline{\mathbb{R}} = [-\infty, \infty]$. We use "$X \in \Sigma$" to mean "$X$ is Σ-measurable" and "$X \in b\Sigma$" to mean "X is bounded and Σ-measurable".

If Γ is a sub-family of Σ, a function $X : S \to \mathbb{R}^d$ is called Γ-*simple* if $X = \sum_{k=1}^{n} c_k 1_{\Lambda_k}$ for some constants c_k in $\mathbb{R}^d$, sets $\Lambda_k \in \Gamma$, and $n \in \mathbb{N}$. Such a function is Σ-measurable. Conversely, any Σ-measurable function is a pointwise limit of a sequence of Σ-simple functions. For example, a Σ-measurable function $X : S \to \mathbb{R}$ is the pointwise limit of the sequence