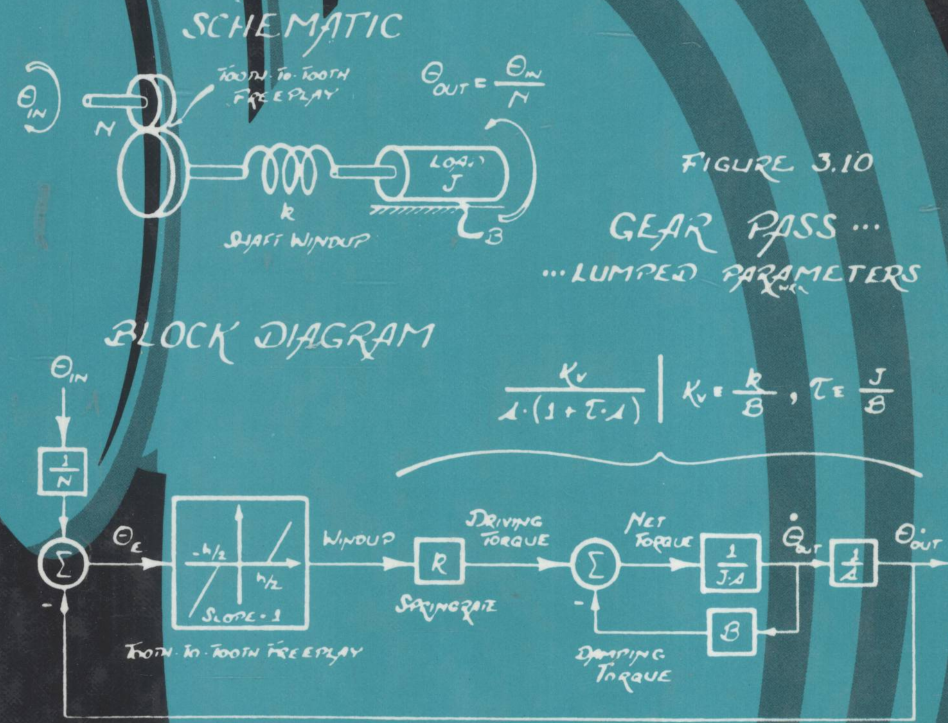
NONLINEAR SYSTEM DYNAMICS



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To Janet . . . sine qua nil.

Dick

To Ellen

Our life together continues to be a series of delightful experiences.

Bob

To Joanne, Kenneth, Beth, Carla, and Andrea

The more you know, the better prepared you are for life's adventures.

Dad

Preface

Engineers, scientists, and applied mathematicians are habitually curious about behavior of physical systems. More often than not they will model the system and then analyze the model, hoping to expose the system's dynamic secrets. Traditionally, linear methods have been the norm and nonlinear effects were only added peripherally. This bias for linear techniques arises from the consummate beauty and order in linear subspaces and the elegance of linear independence is too compelling to be denied. And the bias has been, in the past, fortified by the dearth of nonlinear procedures, rendering the study of nonlinear dynamics untidy. But now a new attractiveness is being conferred on that nondescript patchwork, and the virtue of the hidden surprises is gaining deserved respect.

With a wide variety of individual techniques available, the student and the engineer as well as the scientist and researcher, are faced with an almost overwhelming task of which to use to help achieve an understanding sufficient to reach a satisfying result. If linear analysis predicts system behavior sufficiently close to reality, that is delightful. In the more likely case where nonlinear analysis is required, we believe this text fills an important void.

We have tried to compile and bring some order to a large amount of information and techniques, that although well known, is scattered. We have also extended this knowledge base with new material not previously published. Included in these new procedures and techniques are Chapter 2 (finding analytic solutions to nonlinear differential equations), Chapter 7 (investigations into the stability of digital recursion and understanding the events of numerical analysis which lead to higher-order furcations, chaos, etc.), and Chapter 9 (a tutorial which shows how to use the desk top personal computer to analyze systems).

Finding a closed form solution to nonlinear differential equations has long been a study in frustration. In Chapter 2 we extend the class of equations with known solutions by use of a variety of transformations. Relationships between the known solutions to linear and nonlinear differential equations and those of derivable classes of other nonlinear differential equations are developed yielding a means of solving these new equations.

There is no guarantee that the appropriate transformation exists (it is, after all, nonlinear), so the analyst often turns to digital simulation, in the hope that it will adequately approximate the solution. A tutorial is supplied in Chapter 9 which enables the modeler to simulate his system on a spread-sheet-equipped personal computer. It presents methods of approximating ordinary differential equations with finite difference equations which are then easily recursed to provide solutions which are rapidly plotted.

A topic of increasing interest which naturally evolves from recursions is chaos. In Chapter 7 we have presented some of its aspects, notably the stability of recursions starting with Jacobi's iteration of linear algebraic equations, then the nonlinear Mechanic's rule for extracting square roots, followed by the furcations in the logistics map, and finally, recursions of Lin's method for extracting quadratic factors from polynomials, showing convergence, divergence, furcations and essential chaos.

Just as there is a little order in the nonlinear world, the order of our presentation is not sacrosanct. Having taught this material repeatedly in courses involving Nonlinear Controls or System Dynamics, we are comfortable with this order, but leave it to the reader to select his own. Only an introductory knowledge of differential equations and linear controls is presumed, and, the material can be used in either undergraduate or graduate curricula.

The completion of the manuscript required more sacrifice and pain than we'd ever imagined at its initiation. We could not have finished without the unflinching support of our wives, Janet Kolk and Ellen Lerman, the encouragement and helpful criticism of Dr. Dan DeBra of Stanford University, and Professor George Murphy of Lowell University, nor the unselfish assistance of Mrs. Lee Brown in preparing the manuscript. We thank them sincerely, one and all.

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Linearity—A Shangri-la

I.1 WHY LINEARITY?

Linear theory is a formidable modeling tool which is unmatched by its nonlinear counterpart. While, as engineers, we are fully aware of Mother Nature's ubiquitous nonlinearity, we are nevertheless lured by the power of eigenvalues to describe a system's dynamics. Their compact explanation of an aircraft's Dutch Roll or the attenuation in an operational amplifier is satisfying and conclusive. That they are unique to the linear world and do not exist in Mother Nature's Kingdom is something we tend to overlook, making us susceptible to the illusion that her Kingdom is truly linear. As Arnold Tustin[1] expresses it . . .

> . . . closed sequence systems are delightfully simple to understand and—even more important—very easy to handle in exact mathematical terms. Because of this, most introductory accounts of control systems either brazenly or furtively assume that all such systems are linear . . .''

Linearity invades much of our daily life. The change in our pocket is the same no matter how the coins are counted. If an aspirin is good, then two are better where we vaguely equate better to ''twice-as-good.'' There is, of course, a loose contradiction that the whole bottle can be harmful, but this is not considered a rational option. Real and complex numbers add like change, as do vectors. The world abounds in linear things, but occasionally we encounter unusual quantities like nonplanar Eulerian angles which ''add'' differently, according to their order. The Gas Law is an awkward relation between pressure, density, and temperature, and Maxwell's equations are even more challenging. These are, however, a distinct minority, or so it would seem, because by and large our formal engineering training follows the safe path of linear analysis.

[1]*Automatic Control,* A Scientific American Book, Simon and Schuster, 1955.

1.2 FROM WHENCE LINEARITY

The linear problem has evolved as dominant in both algebra and differential equations because the proposition of linearity bears a myriad of analytic methods that, proof upon proof, has produced a vast body of analysis.

Chronologically, algebra developed before differential equations, yet it is interesting that linear algebra, per se, is a relatively recent development, while linear ordinary differential equations have been studied for 300 years. Linear algebraic equations were not intentionally avoided prior to 1950; on the contrary, most of what then rapidly developed as linear algebra was well-known hundreds of years before, but it resided in that amorphous aggregate known simply as algebra. World War II fostered an interest in mini-max problems, Von Neuman developed game theory, and computers made linear pogramming practical. By then Heaviside's operational calculus had been clothed in the respectability of LaPlace Transforms, which reduced differential equations to polynomials. These factors were recognized by the mathematics community, which responded by sifting out the linear part of what was already available and created linear algebra.

1.3 WHAT IS LINEARITY?

Starting with common addition, a commutative binary rule of combination, any two quantities can be "added." Next, consider a special process, P, which can operate on either or both the quantities x and y. P can be a function, an operation, a transformation, or a mapping, but what makes it special is its capability to distribute over addition,

$$P(x + y) = P(x) + P(y)$$

This defines a linear process, and any process failing the equality is by default nonlinear. It follows that for a, a scalar, and scalar multiplication defined through the continuum of real numbers by successive additions,

$$P(ax) = aP(x)$$

An interesting observation is the provable fact that among all of the real continuous[2] algebraic functions, $f: R \rightarrow R$, imposing linearity on f; that is,

$$f(x + y) = f(x) + f(y) \qquad \text{for } x, y \in R$$

the only thing f can be is scalar multiplication, so,

$$f(x) = kx \quad \text{for } k \in R$$

Thus, the only continuous algebraic function permitted within the orderliness of linearity is a straight line through the origin. The dogma of linearity eschews quadratics, cubics, polynomials in general, and even straight lines with a dc-component. This abstract and perhaps stark approach to linearity points up its rather confining aspect by conceding that a single unique function from among an infinity of applicants is the sole survivor.

A more pleasing approach is usually followed, in which some as yet unknown function of specific variables is hypothesized. With little justification, the function is assumed to have a Taylor series expansion about an operating point (often selected to be the origin), displacements from which are argued to be sufficiently small to truncate their products and powers. The result is identical to the formal imposition of linearity, producing a linear combination of the displacements, which represents the total derivative with partials evaluated at the operating point.

An example is the aerodynamic force on a body which is influenced by many variables, such as air density ρ, speed v, temperature T, viscosity ϑ, Mach number M, and Reynolds number Re, which is expressed as,

$$\text{Force} = f(\rho, v, T, \vartheta, M, Re)$$

The procedure expresses the increment of force in terms of variable increments.

A linear combination of perturbations

$$\begin{aligned}\Delta\text{Force} = {} & (\partial f/\partial \rho)\Delta\rho + (\partial f/\partial v)\Delta v + (\partial f/\partial T)\Delta T + (\partial f/\partial \vartheta)\Delta\vartheta \\ & + (\partial f/\partial M)\Delta M + (\partial f/\partial Re)\Delta Re + \cdots + [\text{Higher order} \\ & \text{terms (HOTs), i.e., products and powers of perturbations}]\end{aligned}$$

where the partials are evaluated as scalars at an operating point, a direct consequence of discarding HOTs.

I.4 THINGS LINEAR OR NOT LINEAR

Typically the linear process, P, is one of the familiar operations, normally occurring in well-behaved engineering problems, a few of which are listed in Table I.1.

TABLE I-1 Some Linear Processes

Scalar multiplication	$a(x + y) = ax + ay$
Differentiation	$d(x + y) = d(x) + d(y)$
Integration	$\int (x + y)\, dt = \int x\, dt + \int y\, dt$
Real (imaginary) part	$Re(x + y) = Re(x) + Re(y)$
Fourier (Laplace) transforms	$F(x + y) = F(x) + F(y)$

[2]Here "continuous" means differentiable and Taylor series-expandable, thus avoiding the undifferentiable functions of Bolzano and Weierstrasse, which have not yet appeared in engineering.

There are more nonlinear than linear processes but their abundance goes unnoticed simply because they are not as easily manipulated. A few are listed in Table I-2.

TABLE I-2 Some Nonlinear Processes

Circular (elliptic) functions	$\text{Sin}\,(x + y) \neq \text{Sin}\,(x) + \text{Sin}\,(y)$
Sign of	$\text{SGN}\,(x + y) \neq \text{SGN}\,(x) + \text{SGN}\,(y)$
Logarithm (exponential)	$\ln\,(x + y) \neq \ln\,(x) + \ln\,(y)$
Powers (roots)	$(x + y)^{\alpha} \neq (x)^{\alpha} + (y)^{\alpha}$
Magnitude (phase)	$\|x + y\| \neq \|x\| + \|y\|$

I.5 OUR SHANGRI-LA

One need not look far for evidence of Mother Nature's nonlinearity. Some of the greatest minds in the physical sciences have ratified that premise with the Gas Law, both Newton's second law and his Law of Gravitation, and Bernoulli's flow equation to name a few. That each expresses a form of nonlinearity is simply Her disregard of conformity.

But what if She were linear? Would it be helpful? As things stand, if our system doesn't work we can blame it on nonlinearity. Were it linear, however, we could analyze it, understand its function, and avoid frustration. And engineering would cease to exist because our Hiltonian Systems would all have known outputs, and malfunctions just couldn't occur.

In the following chapters, we investigate the nonlinear world to introduce a range of techniques which provide comfort, useful tools, and additional insights.

Chapter 1
Four Interesting Equations

1.0 THE EQUATIONS

Four differential equations, (i)–(iv), are presented to show that the consummate uniformity of linear equations is not shared by nonlinear equations

$$\dot{x} + x = 0 \tag{i}$$

$$\dot{x} + |x| = 0 \tag{ii}$$

$$|\dot{x}| + x = 0 \tag{iii}$$

$$|\dot{x}| + |x| = 0 \tag{iv}$$

which is illustrated by addressing three essential features of the solutions.

1. *The Method of Solution:* The solution to (i), while easily found today, was obscure to its discoverer who knew only algebra and differentiation. Take away the number e, integration, the logarithmic and exponential functions, and the method is no longer straightforward. And, whatever method is employed, its existence is implied.
2. *Existence of a Solution:* Whether a solution exists is (or should be) of fundamental importance in engineering. $1 + 2 = 0$ is not an equation but a contradiction, unless the addition is modulo 3. Likewise a differential "equation" can be a disguised but unforgiving contradiction as in (iv). Linearity bestows the guarantee that a solution exists—a fact that is often taken for granted, and inappropriately extended to nonlinear equations.
3. *Uniqueness of a Solution:* If one solution is found, can there be others? Or, is this the only solution? If so, the solution is unique as in (i) and (ii). Should there be more than one, as in (iii), the solutions are not unique. And if there is no solution, the question of uniqueness is moot as in (iv)

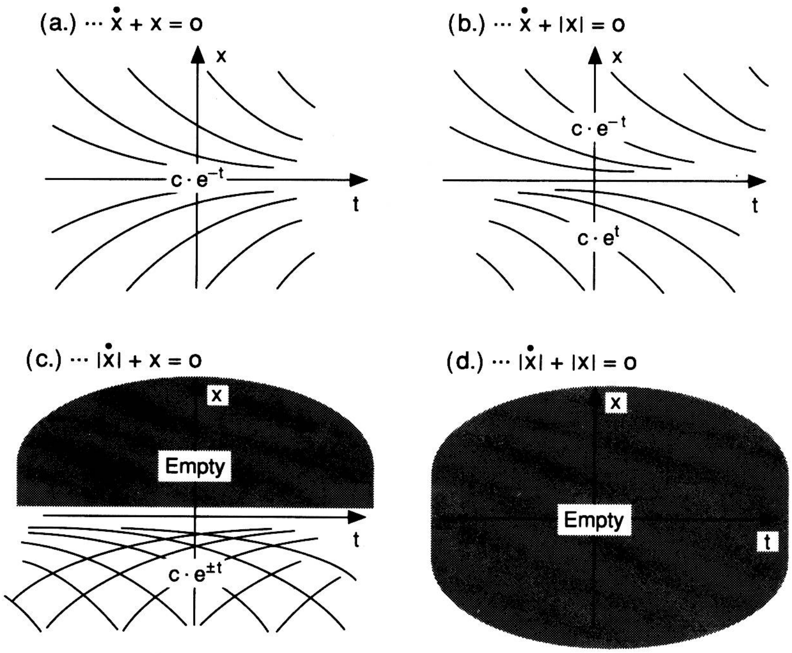

Figure 1-1. Solutions to four equations.

1.1 THE SOLUTIONS—A PREVIEW

The solutions to (i)–(iv) are shown in Fig. 1.1. The unifying relations are the real exponentials, $e^{\pm t}$, one or both of which satisfy the equation if it can be satisfied.

1.2 SOLVING EQUATION (i): $\dot{x} + x = 0$

Four methods of solution are presented, each with its own interpretation. It is helpful to slightly generalize the equation by defining a scaled time, T.

$$T = t/\lambda$$

so $dT = \lambda dT$, and thus $x' + \lambda x = 0$, where $x' = dx/dT$

1.2.1 Method 1—Separating Variables

Ordinary differentials, dx and dT, follow the rules of common algebra, yielding

$$dx/x = -\lambda dt$$

From elementary calculus, the left side integrates to the logarithm, the inverse of which is the exponential.

> *Remark:* We have, here, presumed knowledge of both the logarithm and exponential, as well as integration. Were we ignorant of these (as was the first person to formulate the equation), it would be necessary to develop them from only those facts expressed by the equation.

The equation's *solution* is:

$$\ln x = -\lambda T + c_0$$

where $c_0 \equiv \ln x_0$, or

$$x = e^{-\lambda T + c_0} = ce^{-\lambda T}$$

where $c \equiv x_0 = e^{c_0}$.

This is plotted for $\lambda = +1$ [Figure 1.1(a)], where the entire solution plane x vs. t or x vs. T is included in the set of all solutions. There are no regions omitted, and so we confidently assert that the solution exists everywhere.

But is there another solution which the method neglected to expose? The general question of existence and the related question of uniqueness are discussed in Section 1.7, where it is established that this solution passes the sufficiency tests for existence and uniqueness throughout the entire x–T plane. A property of all linear ordinary differential equations with constant coefficients is the existence and uniqueness of solutions, making them a special target of analysis.

1.2.2 Method 2—Identity Under Differentiation

If in the generalized form of equation (i), we choose $\lambda = -1$, then $x' = x$, which asks for a function that reproduces itself under differentiation. Such a property is displayed by the exponential function, e^T.

It is noteworthy that this function can be designated as the operation which inverts the logarithm. That it is a shorthand notation for an infinite series is unnecessary to the solution of the equation. The important point is to recognize that differentiation is the exponential function's identity element.

1.2.3 Method 3—Recursive Approximation

Following Polya's prompt, we guess the answer. Thereafter the guess is refined using the equation to produce a better guess. Hopefully, successive iterations