Compressible Flow

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Preface

The present work had its origin in the one-year graduate course in compressible flow which the author has taught for a number of years at the University of Connecticut. The work is somewhat broader in scope than most works in the field. In addition to the conventional material on perfect gas flow, the book contains chapters on the Navier–Stokes equations, transonic flow, laminar boundary layers, turbulent boundary layers, real gas effects and computational methods.

In the basic theory the approach has been to derive the equations in their most general form first and then to apply the results to particular cases. Thus, the momentum, energy, and continuity equations are first derived for three-dimensional unsteady flow and the results are then reduced to the particular cases of quasi-one-dimensional steady flow, steady two- and three-dimensional flow, unsteady one-dimensional flow, and normal and oblique shocks. In order to make the derivations more readily comprehensible, they are presented in more detail than usual.

Every attempt has been made to discuss the physical significance of the derivations as opposed to a merely formal presentation of the mathematics. Where philosophical or logical questions arise, these are brought out and discussed.

The derivation of the Navier–Stokes equation is presented in some detail in a manner which, it is hoped, will enable the reader to readily grasp the significance of this important equation. In the chapters on transonic flow, laminar boundary layers, and turbulent boundary layers, the most important theories are discussed, at some length, in preference to presenting a cursory overview of as many works as possible. Emphasis is placed on presenting experimental results as well as theoretical results as checks on the validity of the theories. In Chapter 9 the properties of real, as opposed to ideal gases are discussed in detail, as are flows where real gas effects are significant. Finally, a chapter on numerical methods is presented, since numerical analysis has become an indispensable part of the solution of problems in compressible flow.

The present work will appeal mainly to graduate students and workers in the field, but may also be used by advanced undergraduates. A background in vector analysis, ordinary and partial differential equations is desirable as well as a knowledge of elementary fluid mechanics and thermodynamics.

STEFAN SCHREIER

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Contents

1 The Basic Equations

The Nature of the Problem

A compressible fluid differs from an incompressible one in that the density ρ of the fluid is not constant. As a result, an additional unknown is introduced. In incompressible flow it is found that only two natural laws are required for the solution of problems: the law of the conservation of mass and the law of the conservation of momentum. These laws are usually expressed as the continuity equation

$$\nabla \mathbf{V} = 0 \tag{1}$$

and the momentum equation

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p \tag{2}$$

If ρ is known, there are only two unknowns in incompressible flow: the pressure p and the velocity $\mathbf{V}$. Equations (1) and (2) are, therefore, sufficient to render the problem determinate.

In compressible flow, the addition of another unknown, ρ, requires the introduction of some other relation. Such a relation exists in the law of the conservation of energy or the first law of thermodynamics. For closed systems this relation is usually written

$$đQ = dE + đW \tag{3}$$

where $đQ$ represents the heat added to the system, $đW$ the work done by the system, and dE the change in the energy of the system. The bar through the $đ$ in $đW$ and $đQ$ indicates that these quantities are not exact differentials—in other words, Q and W are not properties of the system. The quantity E, in most compressible flow problems, consists of the internal energy E_t and the kinetic energy $\frac{1}{2}mV^2$. Thus, if the heat added and the work done are presumed given, the use of equation (3) adds one new unknown to the system of equations, the internal energy E_t. The internal energy, however, is a state property; that is, an intrinsic property of the gas independent of its history. Pressure, temperature and density, for example, are state properties. Heat and work are not. From elementary thermodynamics we know that for a homogeneous gas, any two state properties are sufficient to determine all the rest. Hence, the internal energy may be related to the pressure and density through an equation of state of the form,

$$E_t = E_t(p, \rho) \tag{4}$$

Thus, through the introduction of equations (3) and (4), the system of equations is once again rendered determinate; that is, there are now four equations and four unknowns: p, ρ, E_t, and $\mathbf{V}$.

The equations cited above apply only to the simplest types of compressible flow problems—those in which the fluid is homogeneous and inviscid. The presence of viscosity and chemical reactions, for example, introduces additional unknowns and renders the problem more complicated. It may thus be seen that problems in compressible flow tend to be more complicated than those in incompressible flow.

Inviscid Compressible Flow—Derivation of the Governing Equations

The equations governing the inviscid compressible flow of a homogeneous fluid were first derived by Euler [1]. Euler considered all the properties of the fluid to be continuous functions of time and space. Consider, for example, the coordinate system shown in Figure 1 and assume a fluid to be flowing through it.

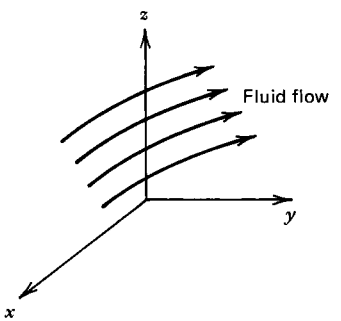

Figure 1. Fluid flow in a Cartesian coordinate system.

Each point in the flow can be described by the values of x, y and z at that point. Under the assumptions of Euler, then, the flow properties may be described by functional relations of the form

$$\mathbf{V} = \mathbf{V}(x,y,z,t), \qquad T = T(x,y,z,t), \qquad \rho = \rho(x,y,z,t)$$

and so on. By filling in numbers for x, y, z, and t, the value of each property at a particular point in space at a particular time is determined.

It should be pointed out that this is not the only possible approach to the problem which may be taken. An alternative would be to describe the properties of the fluid not as functions of a point in space, but as functions of a particular fluid particle. A particle of fluid might be identified, for example, by its position in space at a particular time, say, $t = 0$. At that time the particle may be located in space at a point $x = \xi, y = \eta, z = \zeta$. The properties of the flow may then be described by functional relations of the form

$$\mathbf{V} = \mathbf{V}(\xi,\eta,\zeta,t), \qquad T = T(\xi,\eta,\zeta,t), \qquad \rho = \rho(\xi,\eta,\zeta,t)$$

This is the approach taken by Lagrange [2].

The approach taken by Euler assumes that the fluid is a continuum. Strictly speaking, this assumption is not correct since, according to the molecular theory of matter, a fluid is made up of discrete particles called molecules. Thus, one cannot speak with accuracy about fluid properties at a point. As long as the average distance between molecules is small compared to the scale of the problem, however, this difficulty is more apparent than real. If, for example, we are dealing with the flow over a wing with a ten foot chord, and the average distance between molecules is a millionth of an inch, then, for all practical purposes, the fluid may be considered a continuum. If, on the other hand, the distance between molecules is on the order of a foot, then the situation is no longer so clear. In applying the continuum assumption, we must be careful that the average distance between molecules is small compared to the scale of the problem.

Let us now derive the equations governing the flow of a homogeneous, inviscid, compressible fluid. The first law to be applied is the law of the conservation of mass. This law states that the change in the amount of mass in a given volume over a period of time equals the difference between the amount of mass which has entered the volume during that time, and the amount which has left. Alternatively, one may say that the rate at which the mass in the volume is changing equals the difference between the rate at which mass is entering the volume and the rate at which mass is leaving. One may use the analogy of the bucket with a hole in it (Figure 2). The rate of change of the amount of fluid

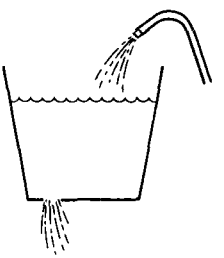

Figure 2. An example of the conservation of mass.

in the bucket is equal to the difference between the rate at which fluid is flowing into the bucket through the hose and the rate at which it is flowing out through the hole in the bottom.

Consider now a fixed volume $\mathcal{V}$ in a cartesian coordinate system through which the fluid is flowing (Figure 3). Let S be the surface of the volume and let $\mathbf{n}$

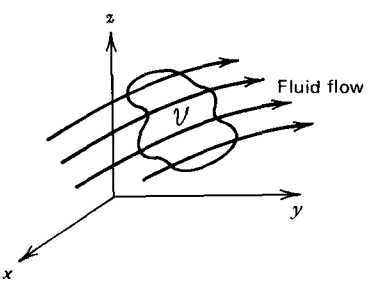

Figure 3. Control volume in three-dimensional flow.

be the unit outward normal at any point of the surface. The net inflow of mass into the volume is then given by,

$$-\int_S \rho \mathbf{V}\cdot\mathbf{n}\,dS$$

where the subscript S indicates that the integral is taken over the surface.

The mass of the fluid in the volume $\mathscr{V}$ at any time is given by

$$\int_{\mathscr{V}} \rho\, d\mathscr{V}$$

where the subscript $\mathscr{V}$ indicates that the integral is taken over the volume.

The rate of change of the mass within the volume is, thus,

$$\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho\, d\mathscr{V}$$

Hence, according to the law of the conservation of mass,

$$\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho\, d\mathscr{V} = -\int_S \rho \mathbf{V}\cdot\mathbf{n}\,dS$$

From vector calculus we know that

$$\int_S \rho \mathbf{V}\cdot\mathbf{n}\,dS = \int_{\mathscr{V}} \nabla(\rho\mathbf{V})\,d\mathscr{V}$$

Also, since $\mathscr{V}$ is not a function of t,

$$\frac{\partial}{\partial t}\int_{\mathscr{V}} \rho\, d\mathscr{V} = \int_{\mathscr{V}} \frac{\partial \rho}{\partial t} d\mathscr{V}$$

Hence,

$$\int_{\mathscr{V}} \frac{\partial \rho}{\partial t} d\mathscr{V} = -\int_{\mathscr{V}} \nabla(\rho\mathscr{V})\,d\mathscr{V}$$

Now since $\mathscr{V}$ is arbitrary, the equation must be valid for any $\mathscr{V}$, no matter how small. This requirement can only be satisfied if the integrands are equal at every point. Thus,

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{V}) = 0 \tag{5}$$

This is the continuity equation for a homogeneous, compressible fluid as it is most commonly written.

We now turn to the second fundamental law, that of the conservation of momentum. This law was first enunciated by Newton [3] and nowadays is most commonly written in the form,

$$\mathbf{F} = m\mathbf{a} \tag{6}$$

As stated in equation (6), this law applies to a closed system. A closed system is one which always consists of the same particles of mass or molecules. In fluid mechanics, however, we are usually interested in open systems. An open system is one into which mass is entering and/or from which mass is leaving. A pipe with water flowing through it would be an open system. We are, thus, faced with the task of restating the law in a form valid for open systems.

The easiest way of applying laws valid only for closed systems to open systems is to apply them to each particle of the open system as it moves along. As long as we stay with the same particle, that particle represents a closed system. If we consider a particle of fluid enclosed by a volume $d\mathcal{V}$, the mass of that particle will be

$$\rho d\mathcal{V}$$

When it comes to expressing the acceleration, however, we encounter a problem in the Eulerian system. As will be recalled, in the Eulerian system, the velocity was expressed as a function of x, y, z, and t.

$$\mathbf{V} = \mathbf{V}(x,y,z,t)$$

This represents a velocity at a particular point in space at a particular time. Thus, the expression

$$\frac{\partial \mathbf{V}}{\partial t}$$

does not represent the acceleration of the particle; rather, it represents the rate of change of the velocity at a particular point in space.

Consider, on the other hand, the Lagrangian system. In this system, the velocity is given by

$$\mathbf{V} = \mathbf{V}(\xi,\eta,\zeta,t)$$

This represents the velocity of a particular particle rather than the velocity at a particular point in space. Hence, the expression

$$\frac{\partial \mathbf{V}}{\partial \mathrm{t}}$$

in Lagrangian coordinates represents the acceleration of a particular particle. It might be considered that because of the ease with which the acceleration may be expressed in the Lagrangian system, this system is preferable to the Eulerian one. In most practical problems, however, other factors, such as the formulation of boundary conditions, enter in, which make the Eulerian system preferable.

Let us, therefore, attempt to express the acceleration of a particle in the Eulerian system. Given that

$$\mathbf{V} = \mathbf{V}(x,y,z,t)$$

it follows that

$$d\mathbf{V} = \frac{\partial \mathbf{V}}{\partial x}dx + \frac{\partial \mathbf{V}}{\partial y}dy + \frac{\partial \mathbf{V}}{\partial z}dz + \frac{\partial \mathbf{V}}{\partial t}dt \tag{7}$$

This represents the difference between the velocity at the point x, y, z at time t and the velocity at the point $x + dx, y + dy, z + dz$ at time $t + dt$. It should be pointed out that this statement is accurate only in the limit as dx, dy, dz, and dt go to zero. Otherwise, it is approximate. Let us now divide both sides of equation (7) by dt. Then,

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \mathbf{V}}{\partial t}$$

Since the ratios of dx, dy, dz, and dt to each other are arbitrary, let us choose them in such a way that the moving particle of fluid located at x, y, z at time t will be located at $x + dx, y + dy, z + dz$ at time $t + dt$. dV will then represent the change in velocity of that particular particle in time dt. If we now take the limit as dx, dy, dz, and dt go to zero, we get

$$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = w$$

where

$$\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

and $d\mathbf{V}/dt$ will then represent the acceleration of a particular particle in the Eulerian system.

$$\frac{d\mathbf{V}}{dt} = u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} + \frac{\partial \mathbf{V}}{\partial t} = \frac{D\mathbf{V}}{Dt} = \mathbf{a} \tag{8}$$

When the particular values

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w$$

are used, as in equation (8), then $d\mathbf{V}/dt$ is usually written

$$\frac{D\mathbf{V}}{Dt}$$

to distinguish it from other values of $d\mathbf{V}/dt$. $D\mathbf{V}/Dt$ is called the substantial derivative.

It may be noted that the value of the operator

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}$$

is not restricted to the velocity. It may be applied to any property of the fluid and will then give the rate of change of that property for a particular particle

of fluid. Thus,

$$\frac{D\rho}{Dt}$$

represents the rate of change of the density of a particular particle, as it moves along,

$$\frac{DT}{Dt}$$

represents the rate of change of the temperature of a particular particle, and so on.

Having expressed the mass and acceleration of a particular particle in the Eulerian system, we may now apply the law of the conservation of momentum, equation (6), to that particle:

$$\mathbf{F} = m\mathbf{a} = \rho d\mathscr{V} \frac{D\mathbf{V}}{Dt} = \mathbf{f} d\mathscr{V} \tag{9}$$

In order to evaluate the force $\mathbf{F}$, it will be easier for us to work in terms of integrals. Let us, therefore, return to the situation shown in Figure 3. It is important to note that the volume $\mathscr{V}$ is fixed in space. Applying equation (9) to each particle of fluid in the volume $\mathscr{V}$ at a given time we have,

$$\int_{\mathscr{V}} \mathbf{f} d\mathscr{V} = \int_{\mathscr{V}} \rho \frac{D\mathbf{V}}{Dt} d\mathscr{V}$$

In the absence of viscous, chemical, electrical, magnetic or body forces on the fluid in volume $\mathscr{V}$, the only force acting on the fluid is that of pressure. Since the pressure of adjacent fluid elements within the volume concels out, only the pressure acting on the surface of the volume $\mathscr{V}$ need be taken into account. If $\mathbf{n}$ is the unit outward normal, the force acting on the volume $\mathscr{V}$ will be given by

$$\int_{\mathscr{V}} f d\mathscr{V} = -\int_S p\mathbf{n}\, dS$$

From vector calculus,

$$\int_S p\mathbf{n}\, dS = \int_{\mathscr{V}} \nabla p\, d\mathscr{V}$$

Hence we have

$$\int_{\mathscr{V}} \rho \frac{D\mathbf{V}}{Dt} d\mathscr{V} = -\int_{\mathscr{V}} \nabla p\, d\mathscr{V}$$

and since this relation must be valid for any volume $\mathscr{V}$ no matter how small,

$$\frac{D\mathbf{V}}{Dt} = -\nabla p \tag{10}$$

This equation is sometimes referred to as the Euler equation.

If a body force $\rho\mathbf{g}$ is present, the total force acting on the volume $\mathscr{V}$ becomes

$$\int_{\mathscr{V}} \mathbf{f}\, d\mathscr{V} = \int_{\mathscr{V}} \nabla p\, d\mathscr{V} + \int_{\mathscr{V}} \rho\mathbf{g}\, d\mathscr{V}$$

and equation (10) becomes,

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho\mathbf{g} \tag{10a}$$

Let us now turn our attention to the third fundamental law, that of the conservation of energy. As pointed out earlier, the first law of thermodynamics may be written for a closed system in the form

$$đQ = dE + đW \tag{11}$$

Once again we must rewrite this law for an open system in order for it to be useful in compressible flow. If the heat transfer $đQ$, the change in energy dE, and the work $đW$ take place in time dt, we may write

$$\frac{dQ}{dt} = \frac{dE}{dt} + \frac{đW}{dt} \tag{12}$$

Let us now take for our system the mass inside volume $\mathscr{V}$ as shown in Figure 3, at some particular time t. If the heat enters or leaves the system only through conduction across its surface S, then we may write, according to Fourier's Law,

$$\frac{đQ}{dt} = -\int_S k\nabla T\cdot\mathbf{n}\, dS \tag{13}$$

where k is the thermal conductivity and $\mathbf{n}$ the unit outward normal on the surface. From vector calculus we know that

$$\int_S k\nabla T\cdot\mathbf{n}\, dS = \int_{\mathscr{V}} \nabla\cdot k\nabla T\, d\mathscr{V} \tag{14}$$

Now let e be the energy per unit mass of the system. Then,

$$\frac{dE}{dt} = \int_{\mathscr{V}} \rho \frac{De}{Dt}\, d\mathscr{V} \tag{15}$$

In the absence of chemical, electrical, viscous, magnetic, or body forces, the work done by the system will be entirely against the surrounding pressure forces. Hence,

$$\frac{đW}{dt} = \int_S p\mathbf{V}\cdot\mathbf{n}\, dS \tag{16}$$

Again, from the divergence theorem,

$$\int_S p\mathbf{V}\cdot\mathbf{n}\, dS = \int_{\mathscr{V}} \nabla(p\mathbf{V})\, d\mathscr{V} \tag{17}$$

Combining equations (12) through (17)

$$-\int_{\mathscr{V}} \nabla\cdot k\nabla T\, d\mathscr{V} = \int_{\mathscr{V}} \rho\frac{De}{Dt}\, d\mathscr{V} + \int_{\mathscr{V}} \nabla(p\mathbf{V})\, d\mathscr{V} \tag{18}$$

Since this equation must be valid for any volume $\mathscr{V}$ no matter how small,

$$-\nabla\cdot k\nabla T = \rho\frac{De}{Dt} + \nabla(p\mathbf{V}) \tag{19}$$

This is one form of the energy equation.

The energy equation is sometimes written in a different form, more convenient for use in some engineering calculations. The energy may be broken up into its component parts for a moving system. For example, if velocity and gravity are present,

$$e = \tfrac{1}{2}V^2 + e_t + gZ \tag{20}$$

where V represents the magnitude of the velocity, e_t the internal energy per unit mass, and gZ the potential energy per unit mass due to gravity. Z represents the distance in the $-\mathbf{g}$ direction. Also,

$$\nabla(p\mathbf{V}) = p(\nabla\cdot\mathbf{V}) + \mathbf{V}\cdot\nabla p$$

and from the momentum equation (10a)

$$\mathbf{V}\cdot\rho\mathbf{g} - \mathbf{V}\cdot\nabla p = \mathbf{V}\cdot\rho\frac{D\mathbf{V}}{Dt} = \rho\frac{D}{Dt}(\tfrac{1}{2}V^2) = -\rho g\frac{DZ}{Dt} - \mathbf{V}\cdot\nabla p$$

Also, we need not restrict heat addition to conduction across the boundary. Heat may also be added or subtracted by chemical reaction. Let us, therefore, indicate the heat added or subtracted per unit volume per unit time by the more general symbol $\dot{q}$. Thus we may write,

$$\dot{q} = \rho\frac{D}{Dt}[e_t + \tfrac{1}{2}V^2 + gZ] + p\nabla\mathbf{V} - \rho\frac{D}{Dt}[\tfrac{1}{2}V^2 + gZ] \tag{21}$$

It is of interest to observe here that the pressure work, $\nabla(pV)$, divides naturally into two parts: $\mathbf{V}\nabla p$, which is the work of increasing or decreasing the kinetic energy of the system, and $p\nabla\mathbf{V}$, which is the work of compressing or expanding the system. This may be illustrated by the use of a somewhat crude example. Consider a slug of gas flowing in a pipe of cross section A:

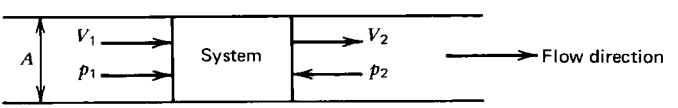

Figure 4. A slug of gas in one-dimensional flow.

In this case,

$$\int_S p\mathbf{V}\cdot\mathbf{n}\,dS = (p_2V_2 - p_1V_1)A \tag{22}$$

The right-hand side of equation (22) may be rewritten

$$(p_2V_2 - p_1V_1)A = A(p_2 - p_1)\left(\frac{V_1 + V_2}{2}\right) + A\left(\frac{p_1 + p_2}{2}\right)(V_1 - V_2) \tag{23}$$

$A(p_2 - p_1)$ represents the net force on the slug, and $(V_1 + V_2)/2$ the average velocity. Hence, the first term on the right-hand side of equation (23) represents the rate of work going to accelerate the slug. $(p_1 + p_2)/2$ represents the average pressure on the slug and $V_1 - V_2$ represents the rate of deformation. Hence, the second term on the right-hand side of equation (23) represents the work going to deform the slug.

Returning now to equation (21), the kinetic and potential energy cancel and we have,

$$\dot{q} = \rho\frac{De_t}{Dt} + p\nabla\mathbf{V} \tag{24}$$

From the equation of continuity,

$$\nabla\mathbf{V} = -\frac{1}{\rho}\frac{D\rho}{Dt}$$

and, adding and subtracting $\rho(D/Dt)(p/\rho)$,

$$p\nabla\mathbf{V} = \frac{p}{\rho}\frac{D\rho}{Dt} + \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) - \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right)$$

$$= -\frac{Dp}{Dt} + \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) \tag{24a}$$

Substituting in equation (24)

$$\dot{q} = \rho\frac{De_t}{Dt} + \rho\frac{D}{Dt}\left(\frac{p}{\rho}\right) - \frac{Dp}{Dt} \tag{25}$$

The specific enthalpy h is defined by

$$h = e_t + \frac{p}{\rho} \tag{26}$$

Hence

$$\dot{q} = \rho\frac{D}{Dt}\left(e_t + \frac{p}{\rho}\right) - \frac{Dp}{Dt}$$

or

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \dot{q} \tag{27}$$

It is in this form that the energy equation is most frequently encountered.

The Navier-Stokes Equation

The equations of Euler, which we have derived above, are deficient in one important respect: they neglect viscosity. This does not mean that they are not useful. In many problems in gas dynamics, the effect of the viscosity is small. However, there are certain cases in which the effect of viscosity is important and hence cannot be neglected. We are thus led to seek an amendation of the Euler equations to include the effects of viscosity. Such an amendation was first made by M. Navier [4] for incompressible flow. In a paper presented to the Royal Academy of Sciences (Paris) on March 18, 1822, Navier pointed out the deficiencies of the equations which Euler had derived sixty-five years earlier, citing as an example the flow from a long-necked vessel which, observation indicated, was considerably slower than predicted by the equations of Euler (the distinction between viscous and inviscid flow was, of course, known to Euler, as well as to d'Alembert twenty years earlier). In order to correct for these discrepancies, Navier proposed a molecular model of matter which is not in accord with modern views on the subject. Nevertheless, using this model, he was able to derive the equations which now bear the name Navier-Stokes equations in the incompressible case, which experiment has proven to be valid and which, still today, may be described as the fundamental equations of fluid dynamics.

In 1845, G. G. Stokes [5] rederived the equations of Navier, using however a macroscopic rather than a molecular hypothesis regarding viscosity, and extending the equations to the case of compressible flow.

Before proceeding to the derivation of the Navier-Stokes equations, as the equations derived by Stokes are now called, let us review the modern theory regarding the origin of viscosity. Viscosity may be defined as that quality of a fluid which makes tangential stresses in the fluid possible. Consider first a two-dimensional flow with a velocity gradient, as shown in Figure 5.

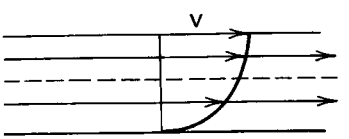

Figure 5. Velocity profile in two-dimensional flow.

The dashed line represents an imaginary dividing line between the upper and lower parts of the fluid. Due to the random motion of the molecules, molecules are always crossing the dashed line from above and below. However, those molecules crossing the line from below will have, on the average, less momentum in the direction of the flow than those crossing the line from above. There is, thus, a net transfer of momentum across the dashed line, the momentum below the dashed line being increased and the momentum above the line being decreased. It follows from Newton's law of motion

$$\mathbf{F} = \frac{d(m\mathbf{V})}{dt} \tag{28}$$

that the fluid above the dashed line is exerting a force on the fluid below, dragging it forward as it were, while the fluid below the dashed line is exerting a force on the fluid above it, tending to retard it.

While the random motion of the molecules explains the presence of tangential stresses in the fluid, it is important to realize that the same random motion may also cause normal stresses. In order to see this let us take the case of a one-dimensional flow with a velocity variation in the direction of the flow, as shown in Figure 6. Let us now examine the flow from the vantage point of an

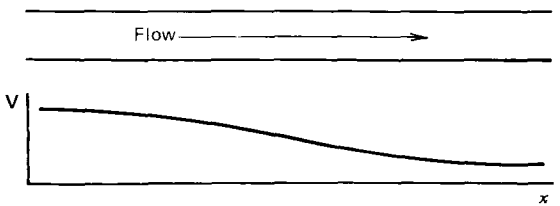

Figure 6. Velocity gradient in the flow direction.

observer traveling with the flow. The observer will see molecules continuously passing from the fluid in front of him to the fluid behind him, and vice versa, due to the random motion of the molecules. However, the molecules passing him from front to back will have less forward momentum than the molecules passing him from back to front. Thus, the fluid in front of the observer is gaining momentum while the fluid behind the observer is losing it. From equation (28), therefore, it is apparent that the fluid in back of the observer is exerting an accelerating force on the fluid in front of him, and the fluid in front of the observer is exerting a retarding force on the fluid in back of him.

It might also be pointed out that the effect of this phenomenon in the present case is to flatten the velocity profile, an effect which is of importance in the study of shock waves.*

Having examined the viscosity qualitatively, we may now inquire into the magnitude of the forces thus produced. It was first proposed by Newton [3] that for the case shown in Figure 5, the magnitude of the viscous force is proportional to the rate of strain. Consider the coordinate system shown in Figure 7.

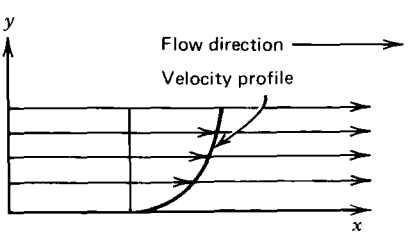

Figure 7. Velocity profile in x-y coordinate system.

*In liquids, intermolecular forces also play an important role in viscosity. In gases, this effect is less important.