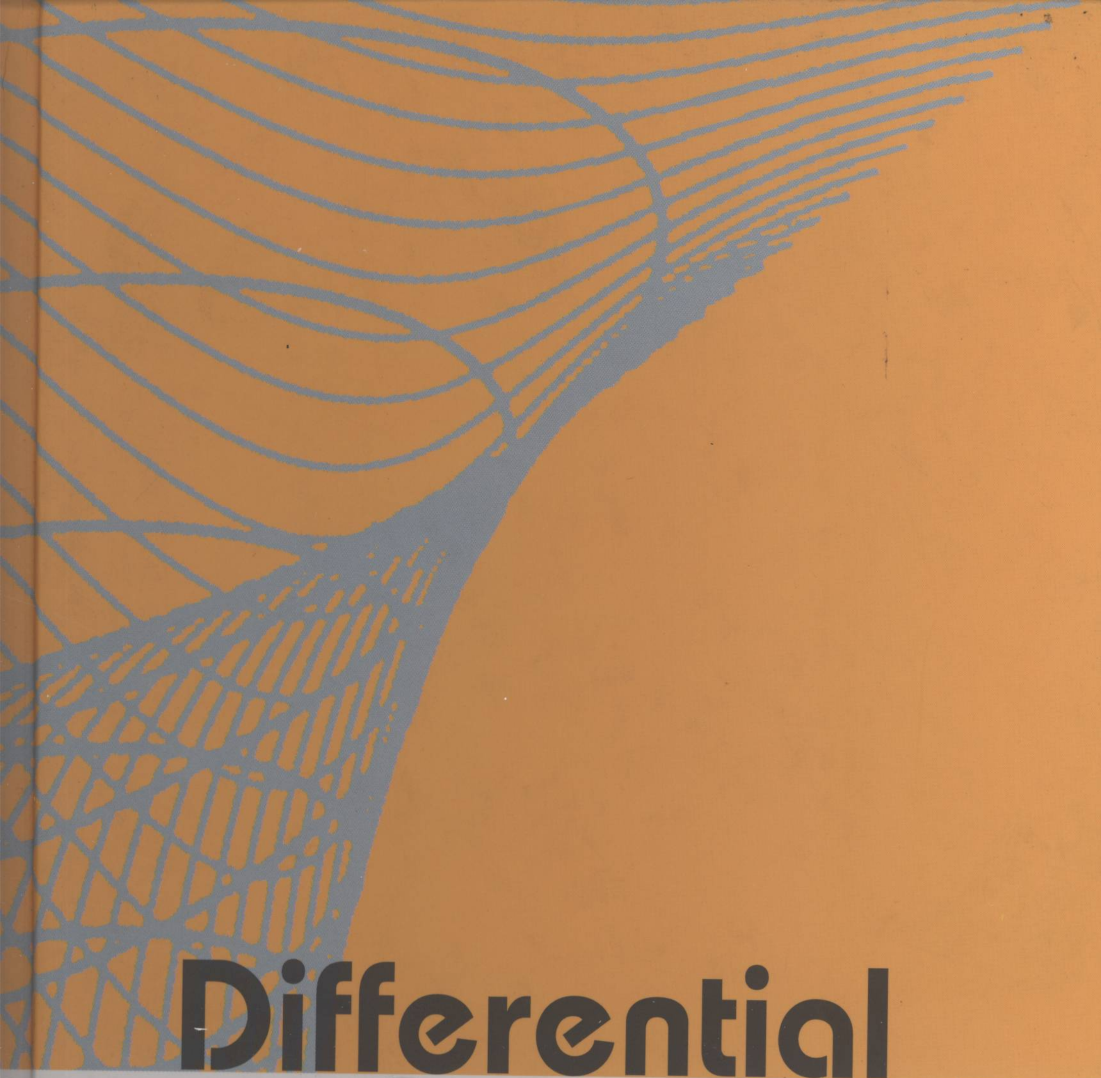
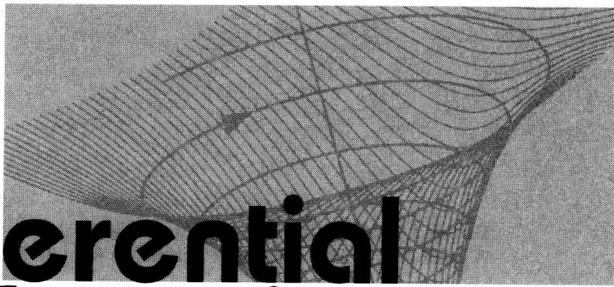


Differential Equations

An Introduction to Basic Concepts, Results and Applications

Ioan I. Vrabie



Differential Equations

An Introduction to Basic Concepts, Results and Applications

Ioan I. Vrabie

"Al. I. Cuza" University of Iaşi and
"O. Mayer" Mathematics Institute of the
Romanian Academy, Iaşi, Romania

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: Suite 202, 1060 Main Street, River Edge, NJ 07661
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

DIFFERENTIAL EQUATIONS
An Introduction to Basic Concepts, Results and Applications

ISBN 981-238-838-9

Printed in Singapore by World Scientific Printers (S) Pte Ltd

Differential Equations

An Introduction to Basic Concepts, Results and Applications

To the Memory of my Parents

Preface

The book is an entirely rewritten English version of the lecture notes of a course on Differential Equations I taught during the last twelve years at the Faculty of Mathematics of "Al. I. Cuza" University of Iaşi. These lecture notes were written in 1999 in Romanian. Their goal was to present in a unitary frame and from a new perspective the main concepts and results belonging to a discipline which, due to the continuous interplay between theory and applications, is by far one of the most fascinating branches of modern mathematics, i.e. *differential equations*. It was my intention to give the reader the opportunity to know a point of view — rather different from the traditional one — offering a possible way to learn differential equations with main emphasis on the Cauchy problem. So, I decided to treat separately the problems of: existence, uniqueness, approximation, continuation of the solutions and, at the same time, to give the simplest possible but complete proofs to some fundamental results which are at the core of the discipline: Peano's local existence theorem, the classification of non-continuable solutions from the viewpoint of their behavior at the end of the existence interval, the continuous dependence of the solution on the data and parameters, *etc.* This goal was by far very hard to accomplish due to the existence of a long list of very good, or even exceptional, textbooks and monographs on this subject covering all levels of difficulty: [Arnold (1974)], [Arrowsmith and Place (1982)], [Barbu (1985)], [Braun (1983)], [Coddington and Levinson (1955)], [Corduneanu (1977)], [Cronin (1980)], [Elsgolts (1980)], [Halanay (1972)], [Hale (1969)], [Hartman (1964)], [Hirsch and Smale (1974)], [Hubbard and West (1995)], [Ionescu (1972)], [Piccinini *et al.* (1984)], [Pontriaghin (1969)], to cite only a few. However, in spite of this challenging competition, I hope that the reader will find this text attractive enough from both the viewpoint of the chosen topics and the presentation.

The book contains a preface, a list of symbols, seven main chapters, a short chapter on auxiliary results, a rather long section including detailed solutions to all exercises and problems, a bibliography and ends with an index. With the sole exception of Chapters 6 and 7, which require some basic results on Lebesgue integral and Measure Theory, it is completely accessible to any reader having satisfactory knowledge of Linear Algebra and Mathematical Analysis. The 36 figures included illustrate the concepts introduced and smooth the way towards a complete understanding of the arguments used in the proofs.

The first chapter starts with a very brief presentation of the main steps made along the last four centuries toward the modern theory of differential equations. It continues with some preliminary notions and results referring to: the concept of solution, some methods of solving elementary equations, various mathematical models described either by differential equations or systems of differential equations, and some basic integral inequalities.

The second chapter contains several fundamental results concerning the Cauchy Problem: the local existence, the continuation of the solutions, the existence of global solutions, the relationship between the local and the global uniqueness, the continuous dependence and the differentiability of the solutions with respect to the data and to the parameters.

The third chapter is merely concerned with some classical facts about the approximation of the solutions: the method of power series, the method of successive approximations, the method of polygonal lines, the implicit Euler method and a particular, and therefore simplified, instance of Crandall–Liggett exponential formula.

In the fourth chapter we apply the previously developed theory to a systematic study of one of the most important class of systems: first-order linear differential systems. Here we present the main results concerning the global existence and uniqueness, the structure of the space of solutions, the fundamental matrix and the Wronskian, the variation of constants formula, the properties of the mapping $t \mapsto e^{t\mathcal{A}}$ and the basic results referring to n^{th}-order linear differential equations.

The fifth chapter is mainly concerned with the study of an extremely important problem of the discipline: the stability of solutions. We introduce four concepts of stability and we successively study the stability of the null solution of linear systems, perturbed systems and fully nonlinear systems respectively, in the last case by means of the Lyapunov's function method. We also include some facts about instability which is responsible for the so-called *unpredictability* and *chaos.*

In the sixth chapter, we start with the study of the concept of prime integral, first for autonomous, and thereafter for non-autonomous systems. Next, with this background material at hand, we present the basic results concerning linear and quasi-linear first-order partial differential equations. Some examples of conservation laws are also included.

The seventh chapter, rather heterogeneous, has a very special character being conceived to help the reader to go deeper within this discipline. So, here, we discuss some concepts and results concerning distributions and solutions in the sense of distributions, Carathéodory solutions, differential inclusions, variational inequalities, viability and invariance and gradient systems.

In the last chapter we include some auxiliary concepts and results needed for a good understanding of some parts of the book: the operator norm of a matrix, compact sets in $C([a,b];\mathbb{R}^n)$, the projection of a point on a convex set.

Each chapter, except that one on Auxiliary Results, ends with a special section containing exercises and problems ranging from extremely simple to challenging ones. The complete proofs of all these are included into a rather developed final section (more than 60 pages).

Acknowledgements. The writing of this book was facilitated by a very careful reading of some parts of the manuscript, by several remarks and suggestions made by Professors Ovidiu Cârjă, Mihai Necula from "Al. I. Cuza" University of Iaşi, by Professors Silvia–Otilia Corduneanu and Silviu Nistor from "Gh. Asachi" Technical University of Iaşi, remarks and suggestions which I took into account. The simplified version of the Frobenius theorem was called to my attention by Dr. Constantin Vârsan, Senior Researcher at The Mathematical Institute of the Romanian Academy in Bucharest. Some of the examples in Physics and Chemistry have been reformulated taking into account the remarks made by Professors Dumitru Luca and Gelu Bourceanu. Professor Constantin Onică had a substantial contribution in solving and correcting most part of the exercises and the problems proposed. A special mention deserves the very careful — and thus critical — reading of the English version by Professor Mircea Bârsan.

It is a great pleasure to express my appreciation to all of them.

Iaşi, November 30^{th}, 2003 *Ioan I. Vrabie*

List of Symbols

$\mathcal{A}^\tau$	—	the transpose of the matrix $\mathcal{A} \in \mathcal{M}_{n\times n}(\mathbb{R})$
$B(\xi, r)$	—	the closed ball centered at ξ of radius $r > 0$
$\overset{\circ}{B}$	—	the interior of the set B
$\mathcal{B}^*$	—	the adjoint of the matrix $\mathcal{B} \in \mathcal{M}_{n\times n}(\mathbb{R})$
$\operatorname{conv} F$	—	the convex hull of F, i.e. the set of all convex combinations of elements in F
$\overline{\operatorname{conv} F}$	—	the closed convex hull of F, i.e. the closure of $\operatorname{conv} F$
$\mathcal{CP}(\mathbb{I}, \Omega, f, a, \xi)$	—	the Cauchy problem $x' = f(t, x)$, $x(a) = \xi$, where $f : \mathbb{I} \times \Omega \to \mathbb{R}^n$, $a \in \mathbb{I}$ and $\xi \in \Omega$
$C([a, b]; \mathbb{R}^n)$	—	the space of continuous functions from $[a, b]$ to $\mathbb{R}^n$
$\mathcal{D}$	—	the data $(\mathbb{I}, \Omega, f, a, \xi)$. So, $\mathcal{CP}(\mathcal{D})$ denotes $\mathcal{CP}(\mathbb{I}, \Omega, f, a, \xi)$
$\operatorname{dist}(\mathcal{K}, \mathcal{F})$	—	the distance between the sets $\mathcal{K}$ and $\mathcal{F}$, i.e. $\operatorname{dist}(\mathcal{K}, \mathcal{F}) = \inf\{\|x - y\|\,;\ x \in \mathcal{K}, y \in \mathcal{F}\}$
$\operatorname{dist}(\eta, \mathcal{K})$	—	the distance between the point η and the set $\mathcal{K}$, i.e. $\operatorname{dist}(\eta, \mathcal{K}) = \inf\{\|\eta - \xi\|\,;\ \xi \in \mathcal{K}\}$
$\mathcal{D}(\mathbb{R})$	—	the space of indefinite differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, with compact support
Δ	—	the compact set $[a, a + h] \times B(\xi, r)$
$\mathcal{D}'(\mathbb{R})$	—	the set of linear continuous functionals from $\mathcal{D}(\mathbb{R})$ to $\mathbb{R}$
$\operatorname{graph}(x)$	—	the graph of $x : \mathbb{I} \to \mathbb{R}^n$, i.e. $\operatorname{graph}(x) = \{(t, x(t))\,;\ t \in \mathbb{I}\}$
$\mathbb{I}$, $\mathbb{J}$, $\mathbb{K}$	—	nonempty intervals in $\mathbb{R}$
$\mathbb{N}$	—	the set of nonnegative integers, i.e. $0, 1, 2, \ldots$
$\mathbb{N}^*$	—	the set of positive integers, i.e. $1, 2, 3, \ldots$
$\nabla_x z$, or ∇z	—	the gradient of the function z with respect to $x_1, x_2, \ldots, x_n$, i.e. $\nabla_x z = \operatorname{grad}_x z = \left(\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \ldots, \frac{\partial z}{\partial x_n}\right)$
$\|\mathcal{A}\|_{\mathcal{M}}$	—	the norm of the matrix $\mathcal{A} \in \mathcal{M}_{n\times m}(\mathbb{R})$, i.e. $\|\mathcal{A}\|_{\mathcal{M}} = \sup\{\|\mathcal{A}x\|_n\,;\ x \in \mathbb{R}^m,\ \|x\|_m \le 1\}$
Ω	—	a nonempty and open subset in $\mathbb{R}^n$
$\mathcal{P}_K(x)$	—	the projection of the vector $x \in \mathbb{R}^n$ on the subset $K \subset \mathbb{R}^n$
$\mathbb{R}$	—	the set of real numbers
$\mathbb{R}^*$	—	the set of real numbers excepting 0
$\mathbb{R}_+$	—	the set of nonnegative real numbers

$\mathbb{R}_+^*$	—	the set of positive real numbers
$\operatorname{supp}\phi$	—	the set $\overline{\{t \in \mathbb{R};\ \phi(t) \neq 0\}}$
Σ	—	a nonempty and locally closed subset in Ω
x'	—	the derivative of the function x
$\dot{x}$	—	the derivative in the sense of distributions of the distribution x
$\mathcal{X}^{-1}(s)$	—	the inverse of the matrix $\mathcal{X}(s)$, i.e. $\mathcal{X}^{-1}(s) = [\mathcal{X}(s)]^{-1}$

Contents

Chapter 1

Generalities

The present chapter serves as an introduction. The first section contains several historical comments, while the second one is dedicated to a general presentation of the discipline. The third section reviews the most representative differential equations which can be solved by elementary methods. In the fourth section we gathered several mathematical models which illustrate the applicative power of the discipline. The fifth section is dedicated to some integral inequalities which will prove useful later, while the last sixth section contains several exercises and problems (whose proofs can be found at the end of the book).

1.1 Brief History

1.1.1 *The Birth of the Discipline*

The name of "*equatio differentialis*" has been used for the first time in 1676 by Gottfried Wilhelm von Leibniz in order to designate the determination of a function to satisfy together with one or more of its derivatives a given relation. This concept arose as a necessity to handle into a unitary and abstract frame a wide variety of problems in Mathematical Analysis and Mathematical Modelling formulated (and some of them even solved) by the middle of the XVII century. One of the first problems belonging to the domain of differential equations is the so-called *problem of inverse tangents* consisting in the determination of a plane curve by knowing the properties of its tangent at any point of it. The first who has tried to reduce this problem to quadratures [1] was Isaac Barrow[2] (1630–1677) who, using a geometric procedure invented by himself (in fact a substitute of the method of separation of variables), has solved several problems of this sort. In 1687 Sir Isaac Newton has integrated a linear differential equation and, in 1694, Jean Bernoulli (1667–1748) has used the *integrand factor method* in order to solve some n^{th}-order linear differential equations. In 1693 Leibniz has employed the substitution $y = tx$ in order to solve homogeneous equations, and, in 1697, Jean Bernoulli has succeeded to integrate the homonymous equation in the particular case of constant coefficients. Eighteen years later, Jacopo Riccati (1676–1754) has presented a procedure of reduction of the order of a second-order differential equation containing only one of the variables and has begun a systematic study of the equation which inherited his name. In 1760 Leonhard Euler (1707–1783) has observed that, whenever a particular solution of the Riccati equation is known, the latter can be reduced, by means of a substitution, to a linear equation. More than this, he has remarked that, if one knows two particular solutions of the same equation, its solving reduces to a single quadrature. By the systematic study of this kind of equation, Euler was one of the first important forerunners of this discipline. It is the merit of Jean le Rond D'Alembert (1717–1783) to have had observed that an n^{th}-order differential equation is equivalent to a system of n first-order differential equations. In 1775 Joseph Louis de Lagrange (1736–1813) has introduced the *variation of constants method*, which, as we can deduce from a letter to Daniel Bernoulli (1700–1782) in 1739, was been already invented by Euler. The equations of the form $Pdx + Qdy + Rdz = 0$ were for a long time considered absurd whenever the left-hand side was not an exact differential, although they were studied by Newton. It was Gaspard Monge (1746–1816) who, in 1787, has given their geometric interpretation and has rehabilitated them in the mathematical world. The notion of *singular solution* was introduced in 1715 by Brook Taylor (1685–1731) and was studied in 1736 by Alexis Clairaut (1713–1765). However, it is the merit of Lagrange who, in 1801, has defined the concept of singular solution in its nowadays acceptation, making a net

[1] By quadrature we mean the method of reducing a given problem to the computation of an integral, defined or not. The name comes from the homonymous procedure, known from the early times of Greek Geometry, which consists in finding the area of a plane figure by constructing, only by means of the ruler and compass, of a square with the same area.

[2] Professor of Sir Isaac Newton (1642–1727), Isaac Barrow is considered one of the forerunners of the Differential Calculus independently invented by two brilliant mathematicians: his former student and Gottfried Wilhelm von Leibniz (1646–1716).

distinction between this kind of solution and that of particular solution. The scientists have realized soon that many classes of differential equations cannot be solved explicitly and therefore they have been led to develop a wide variety of approximating methods, one more effective than another. Newton' statement, in the treatise on *fluxional equations* written in 1671 but published in 1736, that: *all differential equations can be solved by using power series with undetermined coefficients*, has had a deep influence on the mathematical thinking of the XVIII^{th} century. So, in 1768, Euler has imaged such kind of approximation methods based on the development of the solution in power series. It is interesting to notice that, during this research process, Euler has defined the *cylindric functions* which have been baptized subsequently by the name of whom has succeeded to use them very efficiently: the astronomer Friedrich Wilhelm Bessel (1784–1846). We emphasize that, at this stage, the mathematicians have not questioned on the convergence of the power series used, and even less on the existence of the "solution to be approximated".

1.1.2 *Major Themes*

In all what follows we confine ourselves to a very brief presentation of the most important steps in the study of the *initial-value problem*, called also *Cauchy problem.* This consists in the determination of a solution x, of a differential equation $x' = f(t,x)$, which for a preassigned value a of the argument takes a preassigned value ξ, i.e. $x(a) = \xi$. We deliberately do not touch upon some other problems, as for instance the boundary-value problems, very important in fact, but which do not belong to the proposed topic of this book.

As we have already mentioned, the mathematicians have realized soon that many differential equations can not be solved explicitly. This situation has faced them several major, but quite difficult problems which have had to be solved. A problem of this kind consists in finding general sufficient conditions on the data of an initial-value problem in order that this have at least one solution. The first who has established a notable result in this respect was the Baron Augustin Cauchy[3] who, in 1820, has employed the *polygonal lines method* in order to prove the local existence for the initial-value problem associated to a differential equation whose right-hand side

[3]French mathematician (1789–1857). He is the founder of Complex Analysis and the author of the first modern course in Mathematical Analysis (1821). He has observed the link between convergent and fundamental sequences of real numbers.