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S.-T. Yau, Series Editor

Wavelet Analysis and Applications

Proceedings of an International Conference on Wavelet Analysis and Its Applications
November 15–20, 1999
Zhongshan University
Guangzhou, China

Donggao Deng, Daren Huang, Rong-Qing Jia, Wei Lin, and Jianzhong Wang, Editors

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Volume 25

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Wavelet Analysis and Applications

Preface

Wavelet analysis has been one of the major research directions in science in the last decade. More and more, mathematicians and scientists are joining this exciting research area. Certainly, wavelet analysis has had a great impact on areas such as approximation theory, harmonic analysis, and scientific computation. More importantly, wavelet analysis has shown great potential in applications to information technology such as signal processing, image processing, and computer graphics.

China has played a significant role in this development of wavelet analysis as evidenced by many fruitful theoretical results and practical applied projects. In order to exchange ideas and results with international research groups, we organized an international conference on wavelet analysis and its applications during the period of November 15–November 20, 1999, at Zhongshan University, Guangzhou, China. The conference brought together more than 80 researchers in wavelet analysis and related areas from all over the world, including Australia, Canada, China, Germany, Hong Kong, Italy, Japan, Korea, Russia, Singapore, and the United States.

The present proceedings contain twenty-six selective papers from the lectures given in the conference. These papers cover a wide range of research topics of current interest and include many significant results in the study of refinement equations and refinable functions, properties and construction of wavelets, spline wavelets, multi-wavelets, wavelet packets, shift-invariant spaces, approximation schemes and subdivision algorithms, and tilings. Several papers also focus on applications of wavelets to numerical solutions of partial differential equations and integral equations, image processing and facial recognition, computer vision, and feature extraction from data. We thank all the authors for their contributions and believe that these papers will stimulate further interest in wavelet analysis and its applications.

This conference was sponsored by the Ministry of Education of China, the National Science Foundation of China, Nankai University, Peking University, Zhejiang University, and Zhongshan University. We would like to express our sincere thanks to all of the above organizations for their generous financial support to the conference. Special thanks go to our host: the School of Mathematics and Computing Science of Zhongshan University. In particular, we appreciate the efficient work done by Dr. Jianhuang Lai and Ms. Shouzhuang Liang for the conference.

During the process of editing these proceedings, we have received much needed help from Drs. Bin Han and Qingtang Jiang—we are grateful for their assistance. We also thank the publisher, the International Press in Boston, for their cooperation in preparing the manuscript.

Donggao Deng, Daren Huang, Wei Lin
Guangzhou, Guangdong, P. R. China

Rong-Qing Jia
Edmonton, Alberta, Canada

Jianzhong Wang
Huntsville, Texas, USA

December 20, 2000

Participants

A. Aldroubi, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA [aldroubi@math.vanderbilt.edu]

Ryuichi Ashino, Division of Mathematical Sciences, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan [ashino@cc.osaka-kyoiku.ac.jp]

Nin Bi, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China [qsun@math.zju.edu.cn]

K. Bittner, Institute of Biomathematik and Biometrie, GSF National Research Center, Germany [bittner@gsf.de]

G. J. Chae, Division of Applied Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Kusong-Dong Yusong-Gu, Taejon 305-701, Korea

Hanlin Chen, Institute of Mathematics, Academia Sinica, Beijing 100080, P. R. China

Jiming Chen, College of Sciences, Shanghai University, Shanghai 200233, P. R. China

Lizhi Chen, The Seventh Department, National University of Defense Technology, Changsha 410073, P. R. China [lzchen@nudt.edu.cn]

Ting Chen, Department of Communication Project, Shanghai University, Shanghai 201800, P. R. China

Zhongying Chen, Department of Scientific Computing & Computer Applications, Zhongshan University, Guangzhou 510275, P. R. China [insczy@zsu.edu.cn]

Daoqing Dai, Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China [stsddq@zsu.edu.cn]

Donggao Deng, Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China [stsddg@zsu.edu.cn]

Lixin Ding, State Key Lab of Software Engineering, Wuhan University, Wuhan 430072, P. R. China [ding@wipm.whenc.ac.cn]

Guocan Feng, Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China [mcsfgc@zsu.edu.cn]

Xiangchu Feng, Department of Applied Mathematics, Xi'An Elecronic Technology University, Xi'An 710071, P. R. China

Yu Fu, Department of Applied Mathematics, Xi'An Elecronic Technology University, Xi'An 710071, P. R. China [yfu@mail.xidian.edu.cn]

S. S. Goh, Department of Mathematics, National University of Singapore, Singapore 119260 [matgohss@math.nus.edu.sg]

T. N. T. Goodman, Department of Mathematics, University of Dundee, Dundee DD1 4HN, Scotland, U.K. [tgoodman@mcs.dundee.ac.uk]

L. Gori, Dipartimento di Metodi e Modelli Matematici, Università degli Studi di Roma "La Sapienza", Via Antonia Scarpa 16-00161 Roma, Italy [gori@dmmm.uniromal.it]

Jianping Gu, Hubei University, P. R. China

Bin Han, Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544-1000, USA, and Department of Mathematical Sciences, University of Alberta, Edmonton, Canada T6G 2G1 [bhan@ualberta.ca]

Weidong Hao, Department of Mathematics, Shanghai University, Shanghai 201800, P. R. China

D. Hardin, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA [hardin@math.vanderbilt.edu]

Guoen Hu, Department of Applied Mathematics, Institute of Information Engineering, Zhengzhou 450002, Henan, P. R. China

Haiping Hu, Department of Mathematics, Shanghai University, Shanghai 201800, P. R. China

Junquan Hu, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

Daren Huang, Department of Scientific Computation, Zhongshan University, Guangzhou 510275, P. R. China

Rong-Qing Jia, Department of Mathematical Sciences, University of Alberta, Edmonton, Canada T6G 2G1 [jia@xihu.math.ualberta.ca]

Huikun Jiang, Department of Mathematics, Nanjing University, Nanjing 210080, Jiangsu, P. R. China [ata@nju.edu.cn]

Ning Jin, Department of Mathematics, Nanjing University, Nanjing 210080 Jiangsu, P. R. China [njin@nju.edu.cn]

Yuhong Jin, Box 0113 No. 339, JiFang Road, Wuhan 430033, P. R. China

P. Jorgensen, Department of Mathematics, The University of Iowa, Iowa City, IA 52242-1419, USA [pjorgen@blue.weeg.uiowa.edu]

H. O. Kim, Division of Applied Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Kusong-Dong Yusong-Gu, Taejon 305-701, Korea

R. Y. Kim, Division of Applied Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Kusong-Dong Yusong-Gu, Taejon 305-701, Korea

Jianhuang Lai, Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China [stsljh@zsu.edu.cn]

K. S. Lau, Department of Mathematics, Chinese University of Hong Kong, Hong Kong [kslau@math.cuhk.edu.hk]

W. M. Lawton, Department of Mathematics, National University of Singapore, Singapore 119260 [matwml@math.nus.edu.sg]

S. L. Lee, Department of Mathematics, National University of Singapopre, Singapore 119260 [matleesl@math.nus.edu.sg]

Bingzheng Li, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

Dengfeng Li, Southeast University, Nanjing 210096, Jiangsu, P. R. China

Jianping Li, Basic Department of Chongqing Logistics Engineering University, Chongqing 410016, P. R. China [jpli@cqleu.edu.cn]

Song Li, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

Wenming Li, Department of Mathematics, Hebei Normal University, Shijiazhuang 050016, Hebei, P. R. China

Xuezhang Liang, Institute of Mathematics, Jilin University, Changchong 130023, Jilin, P. R. China [xzliang@mail.jiu.edu.cn]

J. K. Lim, Division of Applied Mathematics, Korea Advanced Institute of Science and Technology, 373-1 Kusong-Dong Yusong-Gu, Taejon 305-701, Korea

Wei Lin, Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China [stslw@zsu.edu.cn]

Heping Liu, Department of Mathematics, Peking University, Beijing 100871, P. R. China

Jiufeng Liu, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

Xiaodong Liu, Department of Mathematics, Hong Kong University, Hong Kong

Zhao Liu, Department of Applied Mathematics, North-West Industry University, Xi'an 710071, Shanxi, P. R. China

Zhongxuan Luo, Dalian Science & Engineering University, Dalian 116026, Liaonin, P. R. China

Bolin Ma, Department of Mathematics, Hunan University, Changsha 410082, Hunan, P. R. China [Blma@mail.hun.edu.cn]

C. A. Micchelli, IBM Research Center, P. O. Box 218, Yorktown Heights, NY 10598, USA [cam@watson.ibm.com]

Yongming Pei, Department of Applied Mathematics, North-West Industry University, Xi'an 710071, Shanxi, P. R. China

Lizhong Peng, Department of Mathematics, Peking University, Bejing 100871, P. R. China [lzpeng@pku.edu.cn]

Ruiren Peng, College of Sciences, Shanghai University, Shanghai 200233, P. R. China

F. Pitolli, Dipartimento di Metodi e Modelli Matematici, Università degli Studi di Roma "La Sapienza", Via Antonia Scarpa 16-00161 Roma, Italy [gori@dmmm.uniromal.it]

L. W. Qian, Department of Computational Science, National University of Singapore, Singapore 119260 [qianlw@cz3.nus.edu.sg]

Shuxi Qiu, Department of Mathematics, Xiamen University, Xiamen 361005, Fujian, P. R. China

Qiwen Ran, Department of Mathematics, Harbin Institute of Technology, Harbin 150001, Heilongjiang, P. R. China

T. A. Ridsdill-Smith, Room 4.59 Physics Building, Department of Geology and Geophysics, The University of Western Australia, Nedlands, WA 6907, Australia [tridsdil@geol.uwa.edu.au]

S. D. Riemenschneider, Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA [sherm@math.wvu.edu]

Qiuhui Sheng, Institute of Computational Mathematics and Scientific Engineering Computing, Academia Sinica, Beijing 100080 P. R. China [qhsheng@lsec.cc.ac.cn]

Qingyun Shi, Center of Infofmation Science, Peking University, Beijing 100871, P. R. China

Guoxiang Song, Department of Applied Mathematics, Xi'An Elecronic Technology University, Xi'An 710071, P. R. China

Weifeng Su, Mathematics Institute, Nankai University, Tianjin 300071, P. R. China

Qiyu Sun, Department of Mathematics, National University of Singapore, Singapore 119260 [matsunqy@math.nus.edu.sg]

Wenchang Sun, Mathematics Institute, Nankai University, Tianjin 300071, P. R. China [wenchsun@263.net]

J. S. Tang, Researcher ATR Media Integration & Communications Research Laboratories 2-2, Hikaridai, Seika-cho, Soraku-Gun Kyoto, 619-0288, Japan [stang@mic.atr.co.jp]

Yuan Y. Tang, Department of Computer Science, Hong Kong Baptist University, Kowloon Tong, Hong Kong

Guoqiu Wang, The Seventh Dept., National University of Defense Technology, Changsha 410073, P. R. China [guoqiun@public.cs.hn.cn]

J. Z. Wang, Department of Mathematics, Computer Sciences, and Statistics, Sam Huston State University, Huntsville, TX 77341, USA [mth-jxw@shsu.edu]

Ling Wang, Department of Computing Science, Sichuan Normal University, Chengdu 610066, Sichuan, P. R. China [lwang@sicnu.edu.cn]

Minghui Wang, Department of Mathematics, Peking University, Beijing 100871, P. R. China

Yongge Wang, Department of Mathematics, Peking University, Beijing 100871, P. R. China

Zhenwu Wang, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

Joseph D. Ward, Department of Mathematics, Texas A&M University, College Station, Texas 77843-3368, USA [jward@math.tamu.edu]

Yuchuan Wei, Department of Electrical Engineering, Box 202, Beijing University of Aeronautics and Astronautics, Beijing 100083, P. R. China [yuchuan-wei@21cn.com]

Chenglin Wen, Department of Applied Mathematics, Henan University, Xinxiang 475001, Henan, P. R. China [wencl@mail.henu.edu.cn]

V. Wickerhauser, Department of Mathematics, Washington University, St. Louis, MO 63130, USA [victor@math.wustl.edu]

Shaoquan Wu, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

Yan Wu, AI Lab Institute of Automation Chinese Academy of Sciences, Beijing, P. R. China

Guiwei Xing, Center of Infofmation Science, Peking University, Beijing 100871, P. R. China

Zhengfeng Xiong, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

Chuanxiang Xu, Department of Mathematics, Peking University, Beijing 100871, P. R. China

Huiying Xu, Department of Electronic Projection, Xiamen University, Xiamen 361005, Fujian, P. R. China [zpcai@jingxi'an.xmu.edu.cn]

Shuzhen Xu, Shanghai Jiaotong University, P. R. China

Y. S. Xu, Department of Mathematics, North Dakota Sate University, Fargo, ND 58105, USA [xu@plains.nodak.edu]

I. Novikov, Department of Mathematics, Voronezh State University, 394693 Voronezh, Russia [mail@igorno.vrn.ru]

Jingwen Yan, Department of Electronic Projection, Xiamen University, Xiamen 361005, Fujian, P. R. China [yfwen@jingxi'an.xmu.edu.cn]

Lihua Yang, Department of Scientific Computing & Computer Applications, Zhongshan University, Guangzhou 510275, P. R. China [mcsylh@zsu.edu.cn]

Yirong Ying, Department of Applied Mathematics, Xi'An Electronic Technology University, Xi'An 710071, P. R. China

Xinge You, Hubei University, P. R. China

Zhendong Yuan, Department of Mathematics, East China Normal University, Shanghai 200062, P. R. China [zdyuan@Mathematicsecuu.edu.cn]

Pongchi Yuen, Department of Computer Science, Hong Kong Baptist University, Hong Kong [pcyuen@comp.hkbu.edu.hk]

Xiaoming Zeng, Department of Mathematics, Xiamen University, Xiamen 361005, Fujian, P. R. China [xmzeng@jingxi'an.xmu.edu.cn]

Jingyuan Zhang, Box 503 No. 339, JiFang Road, 430033 Wuhan, P. R. China

Zeyin Zhang, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, P. R. China [yzhang@Mathematicszju.edu.cn]

Ruizhen Zhao, Department of Applied Mathematics, Xi'An Electronic Technology University, Xi'An 710071, P. R. China

Yan Zhao, Department of Applied Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China

D. X. Zhou, Department of Mathematics, City University of Hong Kong, Hong Kong [mazhou@math.cityu.edu.hk]

Xiaolin Zhou, Department of Mathematics, Science College, Xi'an Jiaotong University, Xi'an 710049, Shanxi, P. R. China

Xingwei Zhou, Department of Mathematics, Nankai University, Tianjin 300071, P. R. China [xwzhou@sun.nankai.edu.cn]

Yueping Zhu, Department of Mathematics, Nantong Teachers College, Nantong 226007, P. R. China [zyp@nttc.edu.cn]

Jin Zou, Department of Mathematics, Sichuan Leshan Teachers College, Sichuan, P. R. China

Contents

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Non-uniform Sampling in Multiply Generated Shift-invariant Subspaces of $L^p(\mathbb{R}^d)$

Akram Aldroubi, Qiyu Sun and Wai-Shing Tang

ABSTRACT. Given the samples $\{f(x_j) : j \in J\}$ of a function f belonging to a shift invariant subspace of $L^p(\mathbb{R}^d)$, we give a fast reconstruction algorithm that allows the exact reconstruction of f, as long as the sampling set $X = \{x_j : j \in J\}$ is sufficiently dense.

1. Introduction

In the sampling problem, the objective is to recover a function f on $\mathbb{R}^d$ from its samples $\{f(x_j) : j \in J\}$, where J is a countable indexing set, and where f satisfies some apriori constraints. Typically, the function is assumed to belong to the space of band-limited functions B_Ω, i.e., $\operatorname{supp} \hat{f} \subset [-\Omega, \Omega]$ (see for example [**5, 6, 10, 11**]). For dimension $d = 1$, another characterization states that the space of band-limited functions $B_{[-1/2,1/2]}$ is identical to the space

$$V^2(\mathrm{sinc}) = \Big\{ f = \sum_{k \in \mathbb{Z}} c(k)\,\mathrm{sinc}(x-k) : (c(k)) \in \ell^2 \Big\}. \tag{1.1}$$

Recently, the problem of regular and irregular sampling and reconstruction have been extended to more general shift-invariant spaces of the form

$$V^2(\phi) = \Big\{ f = \sum_{k \in \mathbb{Z}^d} c(k)\phi(x-k) : c \in \ell^2 \Big\} \subset L^2(\mathbb{R}^d) \tag{1.2}$$

(see for instance [**1, 2, 3, 7, 16, 17**]), and weighted shift-invariant spaces of the form

$$V^p_\nu(\phi) = \Big\{ f = \sum_{k \in \mathbb{Z}^d} c(k)\phi(x-k) : c \in {\ell^p}_\nu \Big\} \subset {L^p}_\nu(\mathbb{R}^d) \tag{1.3}$$

[**3**]. The results on irregular sampling and reconstruction as well as their connection to Reproducing Kernel Hilbert Spaces (RKHS) and frames, can be found in [**2**],

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and a complete discussion can be found in [**3**]. Let $\Phi = (\phi_1, \cdots, \phi_r)^T$ and $C = (c_1, \cdots, c_r)^T$. The goal of this article is to discuss the problem of sampling and to extend the results in [**1**] to the case of multiply generated shift-invariant spaces of the form

$$V^p(\Phi) = \Big\{ f = \sum_{k \in \mathbb{Z}^d} C^T(k)\Phi(x-k) : \ C \in (\ell^p)^{(r)} \Big\} \subset L^p(\mathbb{R}^d). \tag{1.4}$$

2. Notation and preliminaries

Let $\mathcal{B}$ be a Banach space, $\mathcal{B}^{(r)}$ will denote r copies $\mathcal{B} \times \cdots \times \mathcal{B}$ of $\mathcal{B}$, together with an appropriate norm $\|\cdot\|_{\mathcal{B}^{(r)}}$ induced by the norm $\|\cdot\|_{\mathcal{B}}$ of $\mathcal{B}$. In particular $L^{p(r)}$, $1 \le p \le \infty$, will denote the space of all measurable vector-valued functions $F = (f_1, \cdots, f_r)^T$ on $\mathbb{R}^d$ with finite norm $\|F\|_{L^{p(r)}}$ given by

$$\begin{aligned} \|F\|_{L^{p(r)}} &= \Big(\sum_{i=1}^r \|f_i\|_{L^p}^p \Big)^{1/p} \quad if\ 1 \le p < \infty, \\ \|F\|_{L^{\infty(r)}} &= \max_{1 \le i \le r} \|f_i\|_{L^\infty} \quad if\ p = \infty. \end{aligned}$$

For $1 \le p \le \infty$, let $(\ell^p)^{(r)}$ be the space of all sequences $D = (d_1, \cdots, d_r)^T$ with finite norms $\|D\|_{(\ell^p)^{(r)}}$, where

$$\begin{aligned} \|D\|_{(\ell^p)^{(r)}} &= \Big(\sum_{i=1}^r \|d_i\|_{\ell^p}^p \Big)^{1/p} \quad \text{if} \quad 1 \le p < \infty, \\ \|D\|_{(\ell^\infty)^{(r)}} &= \max_{1 \le i \le r} \|d_i\|_{\ell^\infty} \quad \text{if} \quad p = \infty. \end{aligned}$$

Let $W(L^p) = W(L^\infty, L^p)$ denote the space of measurable functions f which are "locally in L^∞ and globally in L^p" with finite norm

$$\begin{aligned} \|f\|_{W(L^p)} &= \Big(\sum_{k \in \mathbb{Z}^d} \text{ess sup}_{x \in [0,1]^d} |f(x+k)|^p \Big)^{1/p} < \infty \quad if\ 1 \le p < \infty, \\ \|f\|_{W(L^\infty)} &= \sup_{k \in \mathbb{Z}^d} \ \text{ess sup}_{x \in [0,1]^d} |f(x+k)| < \infty \quad if\ p = \infty. \end{aligned}$$

Endowed with this norm, $W(L^p)$ becomes a Banach space [**8, 9**]. Let $W_0(L^p) = W(C, L^p)$ be the space of all continuous functions in $W(L^p)$. Then it is a closed subspace of $W(L^p)$ and thus also a Banach space [**8, 9**]. We have the following inclusions between the various spaces [**9**]:

$$W_0(L^p) \subset W_0(L^q) \subset L^q \quad \text{for any} \quad 1 \le p \le q \le \infty. \tag{2.1}$$

For $g \in W(L^1)$ and $f \in L^p$, it is not difficult to check that there exists a constant C such that

$$\|f * g\|_{W(L^p)} \le C \|f\|_{L^p} \|g\|_{W(L^1)}.$$

For any vector sequence $D = (d_1, \cdots, d_r)^T \in (\ell^p)^{(r)}$ and $F \in W(L^1)^{(r)}$, define their *semi-convolution* $D *' F$ by

$$D *' F = \sum_{j \in \mathbb{Z}^d} D^T(j) F(\cdot - j).$$