WORLD SCIENTIFIC SERIES ON NONLINEAR SCIENCE
Series A Vol. 13
Series Editor: Leon O. Chua

CHAOS IN NONLINEAR OSCILLATORS

Controlling and Synchronization

M Lakshmanan
K Murali

World Scientific

WORLD SCIENTIFIC SERIES ON NONLINEAR SCIENCE Series A Vol. 13
Series Editor: Leon O. Chua

CHAOS IN NONLINEAR OSCILLATORS

Controlling and Synchronization

M Lakshmanan
K Murali
Bharathidasan University,
Tiruchirapalli, India

World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co Pte Ltd
P O Box 128, Farrer Road, Singapore 912805
USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

CHAOS IN NONLINEAR OSCILLATORS
Controlling and Synchronization

ISBN 981-02-2143-6

Printed in Singapore by Uto-Print

CHAOS IN NONLINEAR OSCILLATORS

Controlling and Synchronization

WORLD SCIENTIFIC SERIES ON NONLINEAR SCIENCE

Editor: Leon O. Chua
University of California, Berkeley

Series A. MONOGRAPHS AND TREATISES

Published Titles

Volume 1: From Order to Chaos
L. P. Kadanoff

Volume 6: Stability, Structures and Chaos in Nonlinear Synchronization Networks
V. S. Afraimovich, V. I. Nekorkin, G. V. Osipov, and V. D. Shalfeev
Edited by *A. V. Gaponov-Grekhov and M. I. Rabinovich*

Volume 7: Smooth Invariant Manifolds and Normal Forms
I. U. Bronstein and A. Ya. Kopanskii

Volume 8: Dynamical Chaos: Models, Experiments, and Applications
V. S. Anishchenko

Volume 9: Frequency-Domain Methods for Nonlinear Analysis: Theory and Applications
G. A. Leonov, D. V. Ponomarenko, and V. B. Smirnova

Volume 12: Attractors of Quasiperiodically Forced Systems
T. Kapitaniak and J. Wojewoda

Volume 14: Impulsive Differential Equations
A. M. Samoilenko and N. A. Perestyuk

Volume 15: One-Dimensional Cellular Automata
B. Voorhees

Volume 16: Turbulence, Strange Attractors and Chaos
D. Ruelle

Volume 17: The Analysis of Complex Nonlinear Mechanical Systems: A Computer Algebra Assisted Approach
M. Lesser

Forthcoming Titles

Volume 18: Wave Propagation in Hydrodynamic Flows
A. L. Fabrikant and Yu. A. Stepanyants

Volume 19: Continuum Mechanics via Problems and Excercises
M. E. Eglit, A. N. Golubiatnikov, D. H. Hodges, J. A. Kamenjargh, V. Karlikov, A. G. Kulikovsky, A. G. Petrov, E. I. Sveshnikova, and I. S. Shikina

Volume 21: Hopf Bifurcation Analysis: A Frequency Domain Approach
G. Chen and J. L. Moiola

Volume 23: From Chaos to Order
G. Chen and X. Dong

PREFACE

It is now well accepted that chaos is an ubiquitous and robust nonlinear phenomenon frequently encountered in nature. During the past two decades or so the concept of chaos has permeated almost all branches of science and engineering. The field is growing into a stage where the initial surprises associated with the phenomenon are waning and new understandings are appearing, while actual controlling and harnessing of it are being contemplated. In these developments, the study of nonlinear oscillators has played a very important role in understanding the chaotic phenomenon. Many ubiquitous systems such as the Duffing oscillator, damped and driven pendulum, driven van der Pol oscillator, and so on have been treated as paradigms in chaos research. The study of such systems is based mostly on approximate analytic approaches and detailed numerical investigations. From another point of view, nonlinear electronic circuits complement these studies through analog simulations. Besides, many new nonlinear electronic circuits have been constructed, which are dynamical systems of interest on their own accord. Chua's diode and related circuits are foremost examples of nonlinear electronic circuits which act as veritable black boxes to study nonlinear phenomena. Thus the twin approaches of numerical analysis and circuit theoretic studies can complement each other in the investigation of bifurcation and chaos phenomena in nonlinear dynamical systems.

In recent times, one has witnessed considerable activity in the controlling of chaotic motions to desired regular orbits, through predetermined small perturbations. Various algorithms have been proposed and implemented successfully to avoid the harmful effects of chaos, when required, and bring back the system to desired regular states by minimal changes. But more surprisingly chaos can also be harnessed in a purposeful way leading to exciting technological applications. Against common beliefs, identical chaotic systems can be synchronized provided an appropriate coupling is introduced between them. This in turn leads to the possibility of spread-spectrum secure communications of both analog and digital signals.

The aim of this book is essentially to analyse the bifurcation and chaos phenomena in typical nonlinear oscillators, especially of damped and driven types, from both dynamical and circuit theoretical points of view, and then to introduce the concept of controlling and synchronization in them. Though many important books on chaotic dynamics have appeared in the recent literature stressing different aspects, the authors believe that the approach taken in this book and the topics covered deal with many aspects not readily discussed in other books on chaos.

Specifically, after giving a brief introduction to the topic of nonlinear dynamics in Chapter 1, we introduce the elementary notions on the dynamics of linear and nonlinear oscillators in Chapter 2. In Chapter 3, a brief introduction to linear and nonlinear circuit theory is provided and the relation to dynamical systems is explained. Bifurcation and chaos phenomena with specific reference to the Duffing oscillator are discussed in Chapters 4 and 5. The different types of attractors, bifurcations, and routes to chaos are discussed in detail for the double-well, single-well and double-hump Duffing oscillators by both numerical analysis and analog circuit simulation in Chapter 4. Complementing these studies, an analytic investigation of the Duffing oscillator is carried out in Chapter 5 through approximate (perturbation and linear stability) analyses, Melnikov criterion and analytic structure (Painlevé singularity structure) studies.

Chapter 6 deals with the bifurcation and chaos aspects of the Bonhoeffer–van der Pol (BVP) and Duffing–van der Pol oscillators, involving numerical, analytical and analog simulation studies. Chapter 7 is devoted to a study of bifurcation and chaos phenomena in nonlinear electronic circuits involving piecewise-linear Chua's diode by both experimental and numerical analyses. We consider the behaviour of Chua's oscillator, the autonomous Chua's circuit, the driven Chua's circuit, the simplest dissipative nonautonomous circuit, and the autonomous Duffing–van der Pol oscillator here.

The final part of the book, consisting of Chapters 8 and 9, deals with some very recent developments in chaotic dynamics, namely controlling of chaos and synchronization of chaotic systems. In Chapter 8 we give a brief account of the various algorithms suggested for controlling of chaos and apply these algorithms to the BVP oscillator as a test case. Some of the methods are also applied to the other oscillators mentioned above. Finally, in Chapter 9 we introduce the Pecora and Carroll method of chaos synchronization with and without cascading, as well as the alternative method of one-way coupling of identical chaotic systems. We then illustrate the possibility of transmitting analog and digital signals using synchronized chaotic signals as carriers in a secure way and apply it to the various oscillators discussed.

The book also contains three appendices on (i) perturbation methods, (ii) van der Pol oscillator, and (iii) some other standard oscillators. Also, a glossary of specialized terms is included.

In the absence of exact analytical methods, numerical studies alone cannot provide a complete picture of the dynamics in the parameter space. For most of the oscillators considered in this book, such phase diagrams given in the text cover only a portion of the parameter space. More extensive analysis is required to cover the entire space of the parameters. However, we do hope that the present book may motivate further work along this direction.

In our endeavour to write this book, we have received whole-hearted support from the members of the Nonlinear Dynamics Group at Bharathidasan University. We have freely used their research results in our discussions in the book. In addition, Dr. M. Daniel and Dr. S. Rajasekar helped us by providing critical comments on the manuscript. We thank them and other members of the group for their cooperation.

In the main body of the book, we have also used materials and figures from many published articles by different authors, which are referred to at appropriate places in the text. We thank the American Institute of Physics and Institute of Physics, U.K., for granting permission to reproduce some of the figures appeared in their journals. For a book of this nature, it is impossible to refer to all the related published literature. Though we have tried to give the relevant references which are familiar to us, we apologize to those authors whose work we did not mention due to either our ignorance or unfamiliarity.

Finally, we wish to record our acknowledgement of the continued support received from Bharathidasan University and the Department of Science and Technology, Government of India, for many years for our various activities in nonlinear dynamics. This support is the main inspiration for us to undertake the endeavour of writing this book.

April 1995
Tiruchirapalli

M. Lakshmanan
K. Murali

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. General

Evolution of physical systems, subject to suitable (constraint-free) internal and external forces and appropriate initial conditions, is often expected to be completely and uniquely determined by Newton's equations of motion (Ref. [1]). For an N-particle system with masses m_i $(i = 1, 2, \ldots, N)$ and forces $\mathbf{F}_i$ acting on them, the dynamics is in general described by the set of second order ordinary differential equations

$$m_i \frac{d^2 \mathbf{r}_i}{dt^2} = \mathbf{F}_i \left(t, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, \frac{d\mathbf{r}_1}{dt}, \frac{d\mathbf{r}_2}{dt}, \ldots, \frac{d\mathbf{r}_n}{dt} \right), \quad i = 1, 2, \ldots, n. \quad (1.1)$$

Here $\mathbf{r}_i$ is the position vector of the ith particle in an inertial frame of reference and Eq. (1.1) is subjected to the prescribed $6N$ initial conditions $\mathbf{r}_i(0)$, $\frac{d\mathbf{r}_i}{dt}|_{t=0}$. It is implicitly assumed here that the initial position and velocity vectors of each particle can be accurately and simultaneously provided. By solving the system of $3N$-second order coupled ordinary differential equations (1.1) along with the initial conditions, one can expect that the future of the system can be completely predicted with any required precision. Such a possibility, in fact, led Laplace to imagine that for a super-intelligence '*nothing could be uncertain and the future, as the past, would be present to its eyes*' (Ref. [2]).

1.2. Nonlinearity and Chaotic Motions

In spite of the impressive conceptual foundation, there are obvious limitations in Newton's description and so in Laplace's dictum:

(i) Presence of external random forces/fluctuations can always introduce a kind of indeterminacy, which is a statistical phenomenon.
(ii) Quantum effects can often lead to indeterminacy, dictated by the Heisenberg's uncertainty relations, due to our limitations in the simultaneous physical measurement of canonically conjugate dynamical variables such as position and momentum.

During recent times, it has been realized that a third kind of limitation can occur in Newton's description of evolution of even simple dynamical systems *when nonlinearity is present* (Refs. [2–10]) in a suitable form. It is true that in order to predict the future behaviour of a physical system accurately, leaving aside the limitations posed by statistical and quantum effects, one has only to solve the initial value problem of a system of deterministic differential equations of the form (1.1).

However, when nonlinear forces are present, the system can in general admit very complex motions and the associated equation of motion cannot in general be exactly integrated. As a result often one has to take recourse to numerical integration of the underlying differential equations. Then any small inaccuracy in the prescription of the initial state or round-off errors at any point or stage of the numerical calculation can build up exponentially fast to make the system deviate appreciably from the actual intended state in a finite time interval. One says that there is an *exponential divergence* of nearby trajectories. There is nothing much we can do about this indeterminism, because however accurate and fast calculating machines we are able to produce, there can still be some small error at some stage of the calculation which will multiply fast in a finite amount of time. This is a fact which we have to live with when nonlinearity is present in an appropriate form.

One might wonder whether the above effect is a mere mathematical or computational artifact or whether it has anything to do with the physical behaviour of the system at all. In fact, one knows now very well that an immediate physical realization of the above exponential divergence of nearby trajectories is the extreme *sensitiveness* of the behaviour of the system on initial conditions. This fact was after all anticipated by H. Poincaré in his celebrated analysis of celestial mechanics (Ref. [2]) itself. Any infinitesimal fluctuation at any time during the evolution of a system can in a finite time lead to a physically realizable effect, the so called 'butterfly effect' as termed by Lorenz (Refs. [2, 4]): "As small a perturbation as a butterfly fluttering its wings somewhere in the Amazons can in a few days time grow into a tornado in Texas".

The above type of complex behaviour admitted by appropriate nonlinear systems, exhibiting extreme sensitiveness to initial conditions, is termed as *chaotic motion* or simply *chaos*, which is a pure manifestation of nonlinearity. Of all possible nonlinear systems, especially of importance are dissipative and conservative systems. There are characteristic differences between the chaos exhibited by these two categories.

1.3. Dissipative and Conservative Nonlinear Systems

(i) *Dissipative systems*: The time evolution of these systems contracts volume in the phase-space (the abstract space of state variables) and consequently the trajectories approach asymptotically either a chaotic or a non-chaotic attractor. The latter may be a fixed point, a periodic limit cycle or a quasiperiodic attractor. These and the chaotic attractors are bounded regions of phase-space towards which the trajectory of the system, represented as a curve, converges in the course of long-time evolution (Refs. [5–10])). Bifurcation or qualitative changes of periodic attractors can occur leading to more complicated and chaotic structures, as a control parameter is varied.

The chaotic attractor is, typically, neither a point nor a curve but a geometrical structure having a self-similar and fractal (often multifractal) nature. Such chaotic attractors are called *strange attractors*. Many physically and biologically important nonlinear dissipative systems, both in low and high dimensions, exhibit strange attractors and chaotic motions. Typical examples are the various damped and driven nonlinear oscillators (Refs. [5–16]), the Lorenz system (Ref. [4]), the Brusselator model (Refs. [13, 14]), the Bonhoeffer–van der Pol oscillator (Ref. [17]), the piecewise linear electronic circuits (Refs. [18–20]), and so on.

(ii) *Conservative or Hamiltonian systems*: Nonlinear systems of *conservative* or *Hamiltonian type* also often exhibit chaotic motions (Refs. [21–23]). But here the phase-space volume is conserved and so no strange attractor is exhibited. Instead, chaotic orbits tend to visit all parts of a subspace of the phase-space uniformly. The dynamics of a nonintegrable conservative system is typically neither entirely regular nor entirely irregular, but the phase-space consists of a complicated mixture of regular and irregular components. In the regular region the motion is quasiperiodic and the orbits lie on tori while in the irregular regions the motion appears to be chaotic but they are not attractive in nature. Typical examples include coupled nonlinear oscillators, the Henon–Heiles system, the anisotropic Kepler problem, and so on. Similarly, the quantum manifestations of such Hamiltonian chaos, namely quantum chaos (Refs. [21–24]), are also of great physical relevance. However, this book does not deal with the Hamiltonian chaos aspects but concentrates only on dissipative systems.

It should be emphasized here that not every nonlinear dynamical system as a rule exhibits chaotic motions. Even very complicated nonlinear systems can sometimes exhibit very coherent and ordered structures such as solitons, dromions, instantons, and so on (Refs. [25, 26]). When a given nonlinear dynamical system will exhibit chaotic behaviour and when it will admit coherent and ordered behaviour are intricate mathematical problems, the understanding of which will constitute an important area of future investigations in the field. Some possible lines of thinking include the Painlevé singularity structure analysis (Refs. [27, 28]), investigation of generalized symmetries (Refs. [28–31]), Melnikov analysis (Refs. [10, 32]) and so on.

1.4. Bifurcations and Chaos-Controlling and Synchronization

In this book we will concentrate mainly on the chaotic motions exhibited by damped and driven nonlinear oscillator systems of interest in different fields of research and will illustrate the rich variety of bifurcations and chaos phenomenon exhibited by them. We will then also discuss how chaos can be controlled to regular motion by minimal efforts and finally the possible technological applications of it in secure communications through the concept of chaos synchronization. As a prelude to these developments we will first consider the oscillations of simple linear and nonlinear systems in the next Chapter.

CHAPTER 2

LINEAR AND NONLINEAR OSCILLATORS

The superposition principle which is valid for linear differential equations is no longer valid for nonlinear ones. A physical consequence is that the frequency of oscillation is in general amplitude-dependent in the case of nonlinear systems, while it is not so in the case of linear systems. Particularly, this can have dramatic consequences in the case of forced and damped nonlinear oscillators, leading to nonlinear resonance and jump (hysteresis) phenomenon for low strengths of nonlinearity parameters. Such behaviours can be analyzed using various perturbation methods. However, as the control parameter varies, the nonlinear systems can enter into more complex motions through different routes, where detailed numerical analysis and possible analog simulations using electronic circuits can be of much help. We will briefly introduce these ideas in the present and next Chapters, while more exhaustive studies will be taken up in the later Chapters. Before discussing the nature of nonlinear oscillations, we will first briefly discuss the salient features associated with a damped and driven linear oscillator in order to compare its properties with nonlinear oscillators.

2.1. Linear Oscillators and Predictability

Physical systems whose motion is described by linear differential equations are called linear systems. If they are associated with oscillatory behaviour, then they are designated as *linear oscillators* (Ref. [33]). The characteristic features of such linear systems are their insensitiveness to infinitesimal changes in initial conditions and the (at the most) constant separation of nearby trajectories in phase space. As a consequence the future behaviour becomes completely pre-