PARTIAL DIFFERENTIAL EQUATIONS for Scientists & Engineers

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Partial Differential Equations for Scientists and Engineers

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for
Scientists and
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Preface

Over the last few years, there has been a significant increase in the number of students studying partial differential equations at the undergraduate level, and many of these students have come from areas other than mathematics, where intuition rather than mathematical rigor is emphasized. In writing *Partial Differential Equations for Scientists and Engineers*, I have tried to stimulate intuitive thinking, while, at the same time, not losing too much mathematical accuracy. At one extreme, it is possible to approach the subject on a high mathematical epsilon-delta level, which generally results in many undergraduate students not knowing what's going on. At the other extreme, it is possible to wave away all the subtleties until neither the student *nor the teacher knows what's going on.* I have tried to steer the mathematical thinking somewhere between these two extremes.

Partial Differential Equations for Scientists and Engineers evolved from a set of lecture notes I have been preparing for the last five years. It is an unconventional text in one regard: It is organized in 47 semi-independent lessons in contrast to the more usual chapter-by-chapter approach.

Separation of variables and integral transforms are the two most important analytic tools discussed. Several nonstandard topics, such as Monte Carlo methods, calculus of variations, control theory, potential theory, and integral equations, are also discussed because most students will eventually come across these subjects at some time in their studies. Unless they study these topics here, they will probably never study them formally.

This book can be used for a one- or two-semester course at the junior or senior level. It assumes only a knowledge of differential and integral calculus and ordinary differential equations. Most lessons take either one or two days, so that a typical one-semester syllabus would be: Lessons 1–13, 15–17, 19–20, 22–23, 25–27, 30–32, 37–39. All 47 lessons can easily be covered in two semesters, with plenty of time to work problems.

The author wishes to thank the editors at Wiley for their invitation to write this book as well as the reviewers, Professor Chris Rorres and Professor M. Kursheed Ali, who helped me greatly with their suggestions. Any further suggestions for improvement of this book, either from students or teachers, would be greatly appreciated. Thanks also to Dorothy, Susan, Alexander, and Daisy Farlow.

Stanley J. Farlow

Partial Differential Equations
for
Scientists and Engineers

Laplacian in Different Coordinate Systems

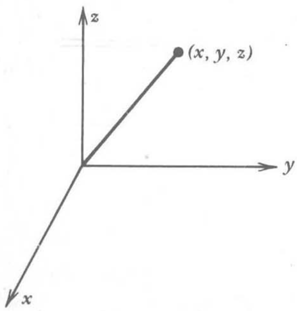
$\nabla^2 u = u_{xx} + u_{yy}$ Two dimensional cartesian Laplacian

$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$ Two dimensional polar Laplacian

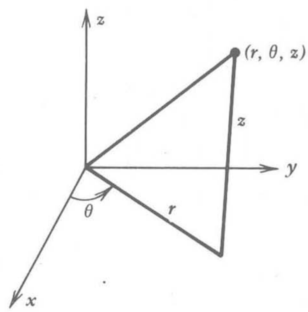
$\nabla^2 u = u_{xx} + u_{yy} + u_{zz}$ Three dimensional cartesian Laplacian

$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}$ Three dimensional cylindrical Laplacian

$\nabla^2 u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + \frac{\cot\theta}{r^2} u_\theta + \frac{1}{r^2 \sin\theta} u_{\phi\phi} = 0$ Three dimensional spherical Laplacian



Cartesian coordinate system



Cylindrical coordinate system

$r \geq 0$

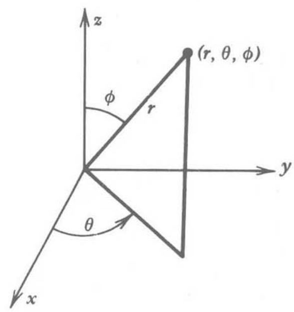
$0 \leq \theta < 2\pi$

$-\propto < z < \propto$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$



Spherical coordinate system

$r \geq 0$

$0 \leq \theta < 2\pi$

$0 \leq \phi \leq \pi$

$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

Elliptic Partial Differential Equations

$\nabla^2 u = 0$ Laplace's equation

$\nabla^2 u + \lambda^2 u = 0$ Helmholtz's equation

$\nabla^2 u = k$ Poisson's equation

$\nabla^2 u + k(E - V)u = 0$ Schrödinger's equation

Hyperbolic Partial Differential Equations

$u_{tt} = c^2 u_{xx}$ One dimensional vibrating string

$u_{tt} = c^2 u_{xx} - hu_t$ Vibrating string with friction

$u_{tt} = c^2 u_{xx} - hu_t - ku$ Transmission line equation

$u_{tt} = c^2 u_{xx} + f(x,t)$ Wave equation with forced vibrations

$u_{tt} = c^2 \nabla^2 u$ Wave equation in higher dimensions

$u_{tt} = c^2 \nabla^2 u - hu_t$ Wave equation with friction

Parabolic Partial Differential Equations

$u_t = \alpha^2 u_{xx}$ One-dimensional diffusion equation

$u_t = \alpha^2 u_{xx} - hu_x$ Diffusion-convection equation

$u_t = \alpha^2 u_{xx} - ku$ Diffusion with lateral heat-concentration loss

$u_t = \alpha^2 u_{xx} + f(x,t)$ Diffusion with heat source (or loss)

Exponential Fourier Transform

$f(x) = \mathcal{F}^{-1}[F] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega$	$F(\omega) = \mathcal{F}[f] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
1. $f'(x)$	$i\omega F(\omega)$
2. $f''(x)$	$-\omega^2 F(\omega)$
3. $f^n(x)$ (nth derivative)	$(i\omega)^n F(\omega)$
4. $f(ax)$ $a > 0$	$\frac{1}{a}F\left(\frac{\omega}{a}\right)$
5. $f(x - a)$	$e^{-ia\omega} F(\omega)$
6. $e^{-a^2x^2}$	$\frac{1}{a\sqrt{2}} e^{-\omega^2/4a^2}$
7. $e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2 + \omega^2}$
8. 1 $\|x\| < a$; 0 $\|x\| > a$	$\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$
9. 1 $\|x\| < 1$; 0 $\|x\| > 1$	$\sqrt{\frac{2}{\pi}}\frac{i}{\omega}(\cos\omega - \frac{1}{\omega}\sin\omega)$
10. $\delta(x - a)$	$\frac{1}{\sqrt{2\pi}}e^{-ia\omega}$
11. $f(x) * g(x)$	$\frac{1}{\sqrt{2\pi}}F(\omega)G(\omega)$
12. $(1 + x^2)^{-1}$	$\sqrt{\frac{\pi}{2}}e^{-\|\omega\|}$
13. $xe^{-a\|x\|}$ $a > 0$	$-2\sqrt{\frac{2}{\pi}}\frac{ia\omega}{(\omega^2 + a^2)^2}$
14. $H(x + a) - H(x - a)$	$\sqrt{\frac{2}{\pi}}\frac{\sin(a\omega)}{\omega}$
15. $\frac{a}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}}e^{-a\|\omega\|}$
16. $\frac{2ax}{(x^2 + a^2)^2}$	$-i\sqrt{\frac{\pi}{2}}\omega e^{-a\|\omega\|}$
17. $\cos(ax)$ $\|x\| < \pi/2a$; 0 $\|x\| > \pi/2a$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2 - \omega^2}\cos(\pi\omega/2a)$
18. $1 - \|x\|$ $\|x\| < 1$; 0 $\|x\| > 1$	$2\sqrt{\frac{2}{\pi}}\left[\frac{\sin(\omega/2)}{\omega}\right]^2$
19. $\cos(ax)$	$\sqrt{\frac{\pi}{2}}[\delta(\omega + a) + \delta(\omega - a)]$
20. $\sin(ax)$	$i\sqrt{\frac{\pi}{2}}[\delta(\omega + a) - \delta(\omega - a)]$

Laplace Transforms

	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
1.	1	$\frac{1}{s} \quad s > 0$
2.	e^{at}	$\frac{1}{s - a} \quad s > a$
3.	$\sin at$	$\frac{a}{s^2 + a^2} \quad s > 0$
4.	$\cos at$	$\frac{s}{s^2 + a^2} \quad s > 0$
5.	$\sinh at$	$\frac{a}{s^2 - a^2} \quad s > \lvert a \rvert$
6.	$\cosh at$	$\frac{s}{s^2 - a^2} \quad s > \lvert a \rvert$
7.	$e^{at} \sin bt$	$\frac{b}{(s - a)^2 + b^2} \quad s > a$
8.	$e^{at} \cos bt$	$\frac{s - a}{(s - a)^2 + b^2} \quad s > a$
9.	$t^n \quad n =$ positive integer	$\frac{n!}{s^{n+1}} \quad s > 0$
10.	$t^n e^{at}$	$\frac{n!}{(s - a)^{n+1}} \quad s > a$
11.	$H(t - a)$	$\frac{e^{-as}}{s} \quad s > 0$
12.	$H(t - a) f(t - a)$	$e^{-as} F(s)$
13.	$e^{at} f(t)$	$F(s - a)$
14.	$f(t) * g(t)$	$F(s) G(s)$
15.	$f^n(t)$ (nth derivative)	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{n-1}(0)$
16.	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right) \quad a > 0$
17.	$\int_0^t f(\tau)\, d\tau$	$\frac{1}{s} F(s)$
18.	$erf(t/2a)$	$\frac{1}{s} e^{a^2 s^2} erfc(as)$
19.	$erfc(a/2\sqrt{t})$	$\frac{1}{s} e^{-a\sqrt{s}}$
20.	$J_0(at)$	$(s^2 + a^2)^{1/2}$

Contents

PART 1

Introduction

LESSON 1

Introduction to Partial Differential Equations

PURPOSE OF LESSON: To show what partial differential equations are, why they are useful, and how they are solved; also included is a brief discussion on how they are classified as various kinds and types. An overview is given of many of the ideas that will be studied in detail later.

Most physical phenomena, whether in the domain of fluid dynamics, electricity, magnetism, mechanics, optics, or heat flow, can be described in general by partial differential equations (PDEs); in fact, most of mathematical physics are PDEs. It's true that simplifications can be made that reduce the equations in question to ordinary differential equations, but, nevertheless, the complete description of these systems resides in the general area of PDEs.

What Are PDEs?

A partial differential equation is an equation that contains partial derivatives. In contrast to ordinary differential equations (ODEs), where the unknown function depends only on *one variable,* in PDEs, the unknown function depends on several variables (like temperature $u(x,t)$ depends both on location x and time t).

Let's list some well-known PDEs; note that for notational simplicity we have called

$$u_t = \frac{\partial u}{\partial t} \qquad u_x = \frac{\partial u}{\partial x} \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad \ldots$$

A Few Well-Known PDEs

$u_t = u_{xx}$ (heat equation in one dimension)

$u_t = u_{xx} + u_{yy}$ (heat equation in two dimensions)

$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (Laplace's equation in polar coordinates)

$$u_{tt} = u_{xx} + u_{yy} + u_{zz} \qquad \text{(wave equation in three dimensions)}$$
$$u_{tt} = u_{xx} + \alpha u_t + \beta u \qquad \text{(telegraph equation)}$$

Note on the Examples

The unknown function u always depends on *more* than one variable. The variable u (which we differentiate) is called the **dependent** variable, whereas the ones we differentiate *with respect to* are called the **independent** variables. For example, it is clear from the equation

$$u_t = u_{xx}$$

that the dependent variable $u(x,t)$ is a function of two independent variables x and t, whereas in the equation

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

$u(r,\theta,t)$ depends on r, θ, and t.

Why Are PDEs Useful?

Most of the **natural laws of physics,** such as Maxwell's equations, Newton's law of cooling, the Navier-Stokes equations, Newton's equations of motion, and Schrodinger's equation of quantum mechanics, are stated (or can be) in terms of PDEs, that is, these laws describe physical phenomena by relating **space and time derivatives.** Derivatives occur in these equations because the derivatives represent *natural things* (like velocity, acceleration, force, friction, flux, current). Hence, we have equations relating partial derivatives of some unknown quantity that we would like to find.

The purpose of this book is to show the reader two things

1. How to *formulate* the PDE from the physical problem (constructing the mathematical model).
2. How to *solve* the PDE (along with initial and boundary conditions).

We wait a few lessons before we start the modeling problem; now, a brief overview on how PDEs are solved.

How Do You Solve a Partial Differential Equation?

This is a good question. It turns out that there is an entire arsenal of methods available to the practitioner; the most important methods are those that change PDEs into ODEs. Ten useful techniques are