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PETER HENRICI

APPLIED AND COMPUTATIONAL COMPLEX ANALYSIS

VOL. 2 SPECIAL FUNCTIONS—INTEGRAL TRANSFORMS—ASYMPTOTICS—CONTINUED FRACTIONS

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VOLUME 2

Special Functions—Integral Transforms —Asymptotics—Continued Fractions

PETER HENRICI

Professor of Mathematics
Eidgenössische Technische Hochschule, Zürich

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PREFACE

In the present Volume II of our three-volume work we continue to discuss algorithmic techniques that can be used to construct either exact or approximate solutions to problems in complex analysis. A focal point for these applications is the evaluation and manipulation of solutions of analytic differential equations. Successive chapters deal with the representation of solutions by (convergent or divergent) series expansions, with the method of integral transforms, with asymptotic analysis, and with the representation of special solutions by continued fractions. The gamma function is dealt with in the opening chapter in the context of product expansions of analytic functions.

Together with its companions, this volume provides a fair amount of information on some of the more important special functions of mathematical physics. However, our treatment of these functions is unconventional in its organization. Whereas the conventional treatment proceeds function by function, giving to each function its due share of series and integral representations, and of asymptotic analysis, our treatment proceeds by general methods and problems rather than by individual functions. Special results thus appear mainly as applications of general principles. The same methodology will be followed in Volume III; for instance, addition theorems will be considered in the context of partial differential equations.

Although I hope that my program has enabled me to illuminate the basic properties of special functions such as the gamma function, the hypergeometric function, the confluent hypergeometric function, and the Bessel functions, it must be pointed out that a full in-depth treatment of any class of special functions was neither intended nor possible. For more detailed information the reader should turn either to specialized treatises or to the monumental Bateman manuscript project (Erdélyi [1953], [1955]), which provides an essentially complete collection of results known up to the early 1950s.

To call this treatment of complex analysis computational is not meant to imply that I deal exhaustively with the problem of obtaining numerical values of a given special function for all possible values of the variable and of the parameters. This topic has grown into a far too specialized and refined science to be treated thoroughly in a book that also must deal with many other topics. The reader is referred to Gautschi [1975] for an excellent survey of the methods that are currently employed. Questions of computational efficiency, including the manipulation of power series, will be dealt with in Chapter 20 (Volume III) of the present work.

The contents of individual chapters are, briefly, as follows. Chapter 8, on infinite products, features, after the necessary preliminaries, some products of importance in number theory, including Jacobi's celebrated triple product identity. The striking combinatorial implications of this identity seem appropriate as an eye-opener to the joys of classical analysis. We then proceed to a standard treatment of the gamma function, proving the equivalence of the definitions by Weierstrass, Gauss, Euler, and Hankel. Stirling's formula is obtained via the Weierstrass definition; derivations from the other three definitions are contained in Chapter 11. The chapter concludes with a discussion of integrals of the Mellin–Barnes type and their application to hypergeometric functions.

The next chapter, on ordinary differential equations, begins with a standard presentation of the analytic theory from the matrix point of view. Here we can apply some of the material given in Chapter 2 on analytic functions with values in a Banach algebra. The treatment of the confluent and of the standard hypergeometric equations is more detailed than is customary in more theoretically oriented texts. In particular we present, on the basis of Riemann's epochal paper [1857], a complete theory of the linear and the quadratic transforms of the hypergeometric series. Because Legendre functions are merely hypergeometric functions permitting quadratic transforms, written in a different notation, we can dispose of these functions very quickly.

Chapter 10, on integral transforms, begins with a broad discussion of the Laplace transform from an elementary point of view, avoiding advanced real variable theory. To present a clean solution of the inversion problem, we provide a self-contained discussion of the Fourier integral theorem (for piece-wise continuous L_1 functions). We next apply the Laplace transform to Dirichlet series and use this opportunity to give a short account of the Riemann zeta function and its connection with the prime number theorem. A presentation of Polya's theory of Laplace transforms of entire functions of exponential type, with its fascinating link between the growth of the original function in a given direction and the location of the singularities of the image function, follows. The next section, on discrete Laplace transforms, contains some generalizations of Polya's theory suggested by the late H. Rutishauser. The chapter concludes with a discussion of the Mellin transform, and of some simple applications of the integral transform idea to problems in mathematical physics.

In Chapter 11, on asymptotics, we have tried, first of all, to give a clear definition of asymptotic series, a concept that is notoriously difficult to absorb for the beginning student. We then prove the important result that the (generally diverging) formal series solutions to differential equations with irregular singular points are asymptotic to appropriate actual solutions. In addition to standard topics, such as Watson's lemma, Laplace's method, the method of steepest descent, Darboux's method, and the Euler–Maclaurin sum formula, we then present some less orthodox subjects such as general asymptotic series (in particular, asymptotic factorial series, for which a useful analog of Watson's lemma is given), and the numerical evaluation of limits by the Romberg algorithm.

The last chapter of this volume, on continued fractions, presented a special challenge to the expositor because the analytical theory of continued fractions is seldom presented in a larger context in a textbook. A novel feature here is the prominence given to Moebius transformations, and with them to the geometric point of view. This not only enables us to deal efficiently with the formal aspects of continued fractions, but also permits us to treat questions of convergence in an intuitively appealing manner. Once again, the *qd* algorithm makes its appearance; here it is used to establish some classical continued fractions representing hypergeometric functions. We then discuss the division algorithm and use it to give an alternate solution of the stability problem for polynomials. The second half of the chapter is devoted to continued fractions of the Stieltjes type. Contrary to other presentations, in which such fractions are merely incidental to a discussion of the moment problem, continued fractions and the functions represented by them here are at the center of interest. Our approach enables us to encompass in a very natural way topics of general interest such as the Stieltjes integral, normal families, Vitali's theorem, and the representation formulas of Herglotz, Hamburger, and Nevanlinna for functions with values in a circular region. Some of these topics will be required again in Volume III. We then proceed to the Carleman convergence criterion and more generally to various estimates for the truncation error, valid also when the corresponding power series has radius of convergence zero. Numerous applications, some of them new, should demonstrate the usefulness of the theory.

As in Volume I, I have restrained myself from using an excess of specialized mathematical notation and terminology to make my subject matter accessible to readers with a variety of backgrounds. Although power series are still favored, this volume can be read without knowing in detail the formal power series approach to complex analysis presented in Volume I. Even within this volume, the chapters are reasonably self-contained to make our text useful also to the casual peruser. Courses of varying length on aspects of applied and computational analysis could be based on almost any combination of chapters; in fact, most of the material was presented in the form of such courses at the ETHZ. By exposing the student to a variety of techniques and applications, we are trying to educate applied mathematicians who are able to contribute to the progress of science by their general expertise as well as by specialized research.

Once more, it is my pleasure to express my thanks to the many individuals who have helped me along in my expository endeavors. In addition to the teachers and collegues mentioned in the preface to Volume I, I wish to record my indebtedness to J.-P. Berrut, P. Geiger, M. Gutknecht, E. Häne, M.-L. Henrici, and J. Waldvogel, who have read parts of the manuscript, corrected errors, and suggested numerous improvements. M. Gutknecht, in addition, wrote the programs for drawing the graphs of the gamma function that appear in Chapter 8. R. Askey and J. F. Kaiser supplied valuable information. R. P. Boas provided not only encouragement but also some important references. During my stay at the Bell Laboratories in 1975, D. D. Warner substantially deepened my understanding of continued fraction theory.

I also wish to express my appreciation to the staff of John Wiley & Sons, who once more handled all problems that arose in the production of a manuscript of mine in the most expert and professional manner.

I dedicate this volume to my wife, who by her optimism and good judgment has been of invaluable help in making the many decisions that were necessary to shape my manuscript into its final form.

PETER HENRICI

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September 1976

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8
INFINITE PRODUCTS

§8.1. DEFINITION AND ELEMENTARY PROPERTIES

Let $\{a_n\}$ be a sequence of complex numbers. It is intuitively clear what is to be understood by the infinite product:

$$\prod_{n=1}^{\infty} a_n = a_1 a_2 a_3 \ldots . \qquad (8.1\text{-}1)$$

We are to form the sequence of partial products $\{p_n\}$, where $p_1 := a_1$, $p_2 := a_1 a_2$, $p_3 := a_1 a_2 a_3, \ldots$. This sequence is somehow to be identified with the infinite product $a_1 a_2 a_3 \ldots$. It is clear, however, that to write down the factors of the product conveys more information than to write down merely the sequence of partial products. If one factor, say a_n, is zero, then all partial products p_m are zero for $m \geqslant n$, and it is impossible to recover the values of the factors a_m from the sequence of partial products for $m > n$. (Contrary to this, the terms of an infinite series can always be recovered from the sequence of its partial sums.) For this reason we shall adopt the following formal definition (see Buck [1965], p. 158):

An ***infinite product*** *is an ordered pair* $[\{a_n\}_1^{\infty}, \{p_n\}_1^{\infty}]$ *of sequences, where* $a_1, a_2, \ldots$ *are complex numbers, and where* $p_n := a_1 a_2 \cdots a_n$, $n = 1, 2, \ldots$.

The numbers a_n and p_n are, respectively, called the nth factor and the nth partial product of the infinite product $[\{a_n\}, \{p_n\}]$. Once this definition is understood, it is completely acceptable to denote an infinite product by a symbol such as (8.1-1), which exhibits only the factors.

Some difficulties also arise if we try to define the concepts of convergence and of value for infinite products. Proceeding as in the case of infinite series, it would be tempting to call the product (8.1-1) convergent if the limit

$$\lim_{n \to \infty} p_n =: p \qquad (8.1\text{-}2)$$

exists, and to define p as the value of the product. In the interest of formulating simple necessary and sufficient conditions for convergence, it is advantageous, however, to call a product of *nonzero* factors convergent only if the limit (8.1-2) exists *and is different from zero.* If a product has zero factors, the limit of its partial products always exists and has the value zero. Convergence would thus not depend on the whole sequence of factors. To avoid this exceptional situation, we call a product with zero factors convergent *if the product of the nonzero factors converges* in the foregoing sense. Thus, in summary, we adopt the following

DEFINITION

The product (8.1-1) *is said to* ***converge*** *if and only if at most a finite number of its factors are zero and if the sequence of partial products formed with the nonzero factors has a limit which is different from zero.*

Let $\prod_{n=1}^{\infty} a_n$ be a convergent infinite product, and let $p_n := a_1 a_2 \cdots a_n$, possible zero factors excluded. Then we have, for n sufficiently large, $a_n = p_n/p_{n-1}$. Because $p_n \to p \neq 0$ there follows

$$\lim_{n\to\infty} a_n = 1. \qquad (8.1\text{-}3)$$

Thus in a convergent infinite product the factors must tend to one. In view of this it is customary to write infinite products in the form

$$\prod_{n=1}^{\infty} (1+a_n),$$

so that $a_n \to 0$ now is a *necessary* condition for convergence. It is easy to see that this condition is not sufficient by considering the example $a_n := 1/n$, $n = 1, 2, \ldots$. Here

$$p_n = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{n}\right) > 1+\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right),$$

and the product is divergent, because the harmonic series is divergent.

The logarithm of a finite product equals the sum of the logarithms of the factors. We thus may expect to derive convergence criteria for products from convergence criteria for sums by taking logarithms. We are led to consider the infinite series

$$\sum_{n=1}^{\infty} \operatorname{Log}(1+a_n), \qquad (8.1\text{-}4)$$

where, for any $z \neq 0$, $\operatorname{Log} z$ denotes the principal value of the logarithm, here defined by the condition $-\pi < \operatorname{Im} \operatorname{Log} z \leq \pi$. Let s_n be the nth partial sum of (8.1-4). Then $p_n = e^{s_n}$, and if $s_n \to s$, it follows from the continuity of the exponential function that $p_n \to p := e^s \neq 0$. Thus the convergence of (8.1-4) is a *sufficient* condition for the convergence of the infinite product.

We now shall show that this condition is also necessary. Suppose that

$p_n \to p \neq 0$. We let $\phi := \operatorname{Im} \operatorname{Log} p$ and define a single-valued branch $\log^* z$ of $\log z$ by the condition $\phi - \pi < \operatorname{Im} \log^* z \leq \phi + \pi$. Then $\log^* z$ is continuous in the vicinity of $z := p$, and there follows

$$\log^* p_n \to \operatorname{Log} p \ (n \to \infty). \tag{8.1-5}$$

We cannot be sure that $s_n = \log^* p_n$ [because the branches of the logarithms in (8.1-5) have already been chosen] but it is certainly true that

$$s_n = \log^* p_n + h_n 2\pi i, \tag{8.1-6}$$

where h_n is some well-determined integer. We wish to show that $\lim_{n\to\infty} s_n$ exists, and in view of (8.1-5) this amounts to showing that $h_n = h_{n-1}$ for all sufficiently large n. Taking the difference of two consecutive terms in (8.1-6), we find

$$\operatorname{Log}(1+a_n) = \log^* p_n - \log^* p_{n-1} + (h_n - h_{n-1}) 2\pi i$$

which we write in the form

$$(h_n - h_{n-1})2\pi i = \operatorname{Log}(1+a_n) + [\log^* p_{n-1} - \operatorname{Log} p] \\ - [\log^* p_n - \operatorname{Log} p].$$

For sufficiently large values of n, $|\operatorname{Im} \operatorname{Log}(1+a_n)| < 2\pi/3$ in view of $a_n \to 0$, and $|\operatorname{Im}(\log^* p_{n-1} - \operatorname{Log} p)| < 2\pi/3$ in view of (8.1-5). Thus ultimately

$$|h_n - h_{n-1}| 2\pi < 2\pi,$$

which implies that $h_n = \text{const}$ and $s_n \to s$, as desired. ■

Altogether we have proved:

THEOREM 8.1a

An infinite product $\prod_{n=1}^{\infty} (1+a_n)$ with nonzero factors converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges, where $b_n := \operatorname{Log}(1+a_n)$ (principal value).

A necessary condition for the convergence of the product $\prod (1+a_n)$ or of the series $\Sigma \operatorname{Log}(1+a_n)$ is that $a_n \to 0$. Now if $a_n \to 0$, $\operatorname{Log}(1+a_n)$ asymptotically behaves like a_n. In fact, from

$$\operatorname{Log}(1+z) = \int_0^z \frac{1}{1+t}\, dt = \int_0^z \left(1 - \frac{t}{1+t}\right) dt$$

we have for $|z| \leq \frac{1}{2}$, integrating along the straight line segment,

$$\operatorname{Log}(1+z) = z(1+wz),$$

where $|w| \leq 1$. Thus if $|a_n| \leq \frac{1}{2}$, then

$$\tfrac{1}{2}|a_n| \leq |\operatorname{Log}(1+a_n)| \leq \tfrac{3}{2}|a_n|. \tag{8.1-7}$$

Hence $\Sigma\,|\mathrm{Log}(1+a_n)|$ converges and diverges simultaneously with $\Sigma\,|a_n|$. An infinite product for which the series $\Sigma\,\mathrm{Log}(1+a_n)$ converges absolutely will be called **absolutely convergent**. In this terminology, we have obtained:

THEOREM 8.1b

A necessary and sufficient condition for the absolute convergence of the product $\prod_{n=1}^{\infty}(1+a_n)$ *is the absolute convergence of the series* $\sum_{n=1}^{\infty} a_n$.

The emphasis here is on absolute convergence. Simple examples (see problems 9 and 10) show that the theorem is not true if the words "absolute convergence" are replaced by "convergence."

These definitions and theorems also apply to the pointwise convergence of infinite products whose factors depend on a variable. A difficulty arises if we wish to define uniform convergence, because of the vanishing of factors. For definiteness, assume that the functions $a_n(z)$ are analytic on a region S, that none of the functions $1+a_n(z)$ vanishes identically on S, and that at most finitely many of these functions assume the value zero on S. The product

$$\prod_{n=1}^{\infty}(1+a_n(z))$$

is said to **converge uniformly** on S if the sequence of partial products formed with those factors that do not vanish on S converges to a limit $\neq 0$ uniformly for all $z \in S$. With this convention, the following analog of Theorem 8.1b holds and is proved similarly:

THEOREM 8.1c

A necessary and sufficient condition for the absolute and uniform convergence of the product $\Pi\,(1+a_n(z))$ *is the absolute and uniform convergence of the series* $\Sigma\,a_n(z)$.

The fundamental theorem on uniformly convergent sequences of analytic functions (Theorem 3.4b) shows that the values of a product of analytic factors that is uniformly convergent on a set S define an analytic function on S, even if the factors with zeros are included.

EXAMPLE

Let $a_n(z) := -z^2/n^2$. We shall show that the product

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) \tag{8.1-8}$$

converges uniformly on every bounded set S. Indeed, let k be an integer such that S is contained in the disk $|z|\le k$. Omitting the first k factors, we obtain the product

$$\prod_{n=k+1}^{\infty}\left(1-\frac{z^2}{n^2}\right),$$

whose factors do not vanish on S. The series

$$\sum_{n=k+1}^{\infty}\left(-\frac{z^2}{n^2}\right)$$

converges uniformly and absolutely on S, because it is majorized by the converging series $k^2\,\Sigma\,(1/n^2)$. Hence the uniform convergence follows by Theorem 8.1c. We conclude that (8.1-8) represents an entire analytic function. Because the zeros of this function are located at $z=\pm1,\pm2,\ldots$, we may expect it to be closely related to $(\pi z)^{-1}\sin(\pi z)$, which has the same zeros and the same value at $z=0$. It is shown in §8.3 that the two functions are, in fact, identical.

PROBLEMS

1. Show that

$$\prod_{n=2}^{\infty}\left(1-\frac{(-1)^n}{n}\right)=\frac{1}{2}.$$

2. In calculus it is shown that

$$\frac{\pi}{2}=\lim_{n\to\infty}\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdots2n\cdot2n}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdots(2n-1)(2n+1)}$$

(Wallis' formula). Show that this may be written

$$\frac{2}{\pi}=\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n)^2}\right).$$

3. Prove that

$$\prod_{n=0}^{\infty}(1+z^{2^n})=\frac{1}{1-z}$$

uniformly on every compact set contained in $|z|<1$.

4. Show that

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

represents an entire analytic function with zeros at the negative integers.

5. Let the real number ξ be given in decimal representation,

$$\xi:=a_{-m}a_{-m+1}\cdots a_0\cdot a_1a_2a_3\cdots.$$

Show that

$$e^{\xi}=\prod_{k=-m}^{\infty}e^{10^{-k}a_k}.$$

6. Show that for any $z \neq 0$,

$$\prod_{k=0}^{\infty} \cos(2^{-k}z) = \frac{\sin 2z}{2z}.$$

[Using trigonometric identities, express the partial products as sums of cosines. Then use the definition of the Riemann integral.]

7. Find the value of Vieta's product,

$$\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots.$$

8. Following D. H. Lehmer (*Amer. Math. Monthly*, 1935), show that the value of the infinite product

$$\left(1-\frac{i}{3}\right)^4\left(1-\frac{i}{17}\right)^4\left(1-\frac{i}{99}\right)^4\left(1-\frac{i}{577}\right)^4\cdots,$$

in which the successive denominators satisfy $d_n = 6d_{n-1} - d_{n-2}$, is purely imaginary.

[Solve the recurrence relation for the d_n. The resulting formula has a meaning for nonintegral n, and there follows

$$\lim_{n\to\infty} \frac{d_{n+1}}{d_n} = 3+\sqrt{8}.$$

Letting $t_n := \tan(\arg p_n^{1/4})$, where p_n is the nth partial product, show that

$$t_n = \frac{d_{n/2+1} - d_{n/2}}{2d_{n/2}}.\Bigg]$$

9. Show that the product

$$\prod_{n=2}^{\infty} \left(1+\frac{(-1)^n}{\sqrt{n}}\right)$$

is divergent, although the series $\Sigma\, (-1)^n/\sqrt{n}$ is convergent.

10. Let

$$a_{2n-1} := -\frac{1}{n^{1/3}},\ a_{2n} := \frac{1}{n^{1/3}-1},\ n = 2, 3, \ldots.$$

Show that the product

$$\prod_{n=3}^{\infty} (1+a_n)$$

is convergent (and, in fact, has the value 1), although the series $\Sigma\, a_n$ is divergent.

§8.2. SOME INFINITE PRODUCTS RELEVANT TO NUMBER THEORY

The main purpose here is the study of certain classical infinite products with variable factors. Although the functions defined by these products do not lie in the mainstream of general complex analysis, some of the identities that exist between them have striking combinatorial and numbertheoretical applications.

The products in question are

$$p(z) := \prod_{n=1}^{\infty} (1+z^n), \qquad q(z) := \prod_{n=1}^{\infty} (1-z^n). \tag{8.2-1}$$

By the criterion of Theorem 8.1c, both products are uniformly and absolutely convergent in any disk $|z| \leq \rho$ where $\rho < 1$. Hence they represent analytic functions that can be expanded in Taylor series for $|z| < 1$:

$$p(z) =: \sum_{n=0}^{\infty} a_n z^n, \qquad q(z) =: \sum_{n=0}^{\infty} b_n z^n. \tag{8.2-2}$$

Because the products contain no zero factors, they are (by the definition of convergence!) different from zero for $|z| < 1$. Thus their reciprocals are likewise analytic for $|z| < 1$; we put, in particular,

$$\frac{1}{q(z)} =: \sum_{n=0}^{\infty} c_n z^n.$$

The coefficients a_n, b_n, c_n can be evaluated very easily. Consider, for example, the nth partial product of $p(z)$,

$$p_n(z) := \prod_{k=1}^{n} (1+z^k).$$

This is a polynomial of degree $\frac{1}{2}n(n+1)$, which we write as

$$p_n(z) =: \sum_{k=0}^{(\infty)} a_k^{(n)} z^k.$$

Because $p_n(z) \to p(z)$ locally uniformly in $|z| < 1$, the basic theorem on convergence of sequences of analytic functions (Theorem 3.4b) implies that for each $k = 0, 1, \ldots,$

$$a_k = \lim_{n \to \infty} a_k^{(n)}.$$

By comparing coefficients of $1, z, \ldots, z^n$ in the relation

$$p_{n+1}(z) = (1+z^{n+1}) p_n(z)$$