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Erich Martensen (Eds.)

Dynamical Problems in Mathematical Physics

PETER LANG

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PREFACE

The 10th conference on "Methoden und Verfahren der mathematischen Physik" was held on February 21-27, 1982 at the "Mathematisches Forschungsinstitut Oberwolfach". The conference was organized by Bruno Brosowski (Frankfurt a.M.) and Erich Martensen (Karlsruhe). Thirty-six papers were presented and approximately 40 people attended from the following countries: Austria, Brazil, ČSSR, Denmark, Great Britain, Hungary, USA, West-Germany, and Yugoslavia.

The purpose of the conference was to apply the greatest possible variety of methods of mathematical physics to a broad scope of concrete problems. In addition, participation by representatives of the various fields of application in physics, engineering sciences, and industry was intended to promote cooperation and mutual stimulation. Timely research results, reported in the lectures, led to lively discussions and fruitful scientific exchanges.

Special attention was devoted to problems of approximation and optimization and to dynamical problems arising in mechanics, fluid dynamics, plasma physics, and scattering theory; hereby questions of mathematical modelling were also treated. A central role was played by analytic and especially functional-analytic methods for partial differential equations and integral equations but there were considered numerical methods, too.

This volume contains the papers on dynamical problems in mathematical physics presented to the conference. The papers on approximation and optimization in mathematical physics appear in volume 27 of this series.

Finally, we would like to thank all those who participated in the conference or contributed to this volume. Thanks are also due to the "Mathematisches Forschungsinstitut Oberwolfach" for financial assistance and for the facilities provided.

August 1982

Bruno Brosowski, Frankfurt a.M.
Erich Martensen, Karlsruhe

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RADAR DETECTION, QUANTUM MECHANICS, AND NILPOTENT HARMONIC ANALYSIS

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ABSTRACT.

As is well known, one of the fundamental tenets of quantum mechanics is the Heisenberg uncertainty principle. "Quantum mechanics" stands here for: The quantum-mechanical description, at a given instant of time, of a finite system of non-relativistic microparticles. According to this principle, not all physical quantities observed in any realizable experiment (even in principle only) can be determined with an arbitrarily high accuracy. There are mutually exclusive ("conjugate") quantities the measurement accuracies of which are interrelated by the uncertainty relationsship. Even under ideal experimental conditions, an increase in the measurement accuracy of one quantity can be achieved only at the expense of decreasing the measurement accuracy of another non-commutating quantity. In the macro-world, there is a similar situation with radar measurements. - It is the purpose of the present paper to show that the real Heisenberg nilpotent group and the symplectic group which forms a subgroup of its automorphism group dominate the radar uncertainty principle according to which there exists an ambiguity in determining the target range and range rate simultaneously.

In order to measure the range of a radar target it is necessary to estimate the time x at which the echo from it arrives at the receiver. If time is counted from the transmission of the radar pulse, the range is $\frac{1}{2}cx$, where c denotes the velocity of electromagnetic radiation. In the case when the radar target is not stationary but is moving relatively to the radar antenna, the carrier frequency of the echo differs from that of the transmitted radar pulse because of the Doppler effect. If we pick the transmitted frequency ω as our natural reference frequency, the Doppler frequency shift is given by

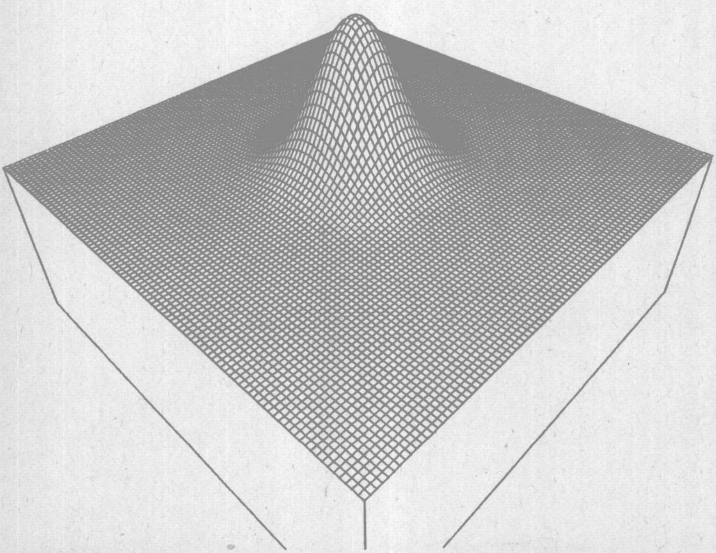
$$y = 2\frac{v}{c}\omega,$$

where v denotes the range rate, i.e., the component of the target velocity in the direction of the radar antenna. For instance, for a target moving at a rate of 500 m.p.h. and for a carrier frequency $\omega = 3000$ MHz, the Doppler frequency shift is $y = 4500$ Hz, an appreciable fraction of the 1 MHz bandwidth typical of radar pulse. Of course, for much larger velocities encountered in satellites the Doppler frequency shift will be even greater.

Whenever it is necessary to distinguish two narrowband signals in the presence of white Gaussian noise, the cross-correlation of the two signals involved plays an important rôle. In radar synthesis the parameters chiefly serving to distinguish two echo signals are their arrival times x and Doppler shifts y of their carrier frequencies from a common reference value as pointed out above. If the signals are assigned the epoch $-(1/2)x$ and $+(1/2)x$ and the carrier frequencies $\omega-(1/2)y$ and $\omega+(1/2)y$, their cross-correlation is termed the radar ambiguity function with respect to the complex envelope $f \in \mathcal{S}(\mathbf{R})$ and can be written (cf. Woodward [6])

$$H(f;x,y) = \int_{\mathbf{R}} f(t+{}^1/_2x)\bar{f}(t-{}^1/_2x)e^{2\pi iyt}dt.$$

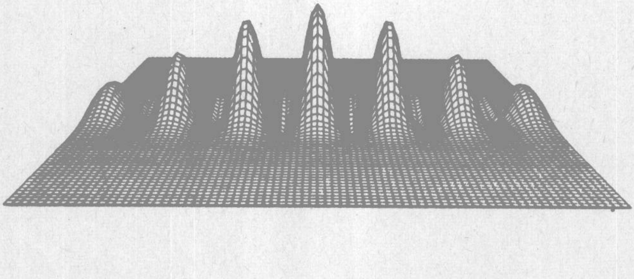
It should be observed that the function $H(f;.,.)$ plays an important rôle in optics as well where it is known as the indeterminacy or spread function. In the case of a Gaussian envelope $f: t \rightsquigarrow a_T e^{-(\frac{t}{T})^2}$ where the constant a_T is determined such that the total energy of the signal equals 1, the radar ambiguity surface, i.e., the set $H(f;\mathbf{R},\mathbf{R})$ admits the following form over the (x,y)- plane:



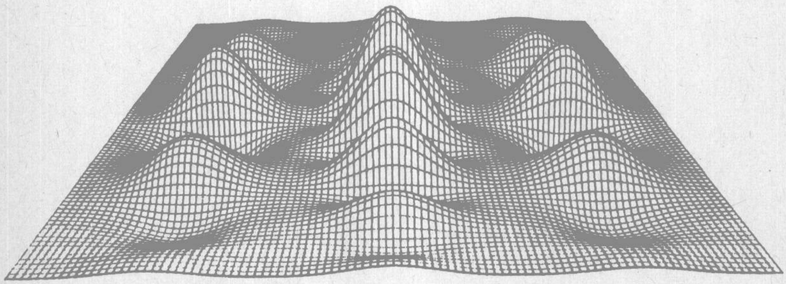
Thus in the case of a single Gaussian pulse the ambiguity surface is peaked merely at the origin (0,0) of the (x,y)-plane and it exhibits no additional peaks elsewhere. However, the range rate v of a target can be measured more accurately if the transmitter sends out a train of coherent radar pulses of the same carrier frequency ω . Therefore, certain radars transmit a sequence of coherent pulses in place of a single pulse. For a repetition period $T_o = 10^{-3}$ sec and a carrier frequency $\omega = 3000$ MHz the ambiguity in range rate

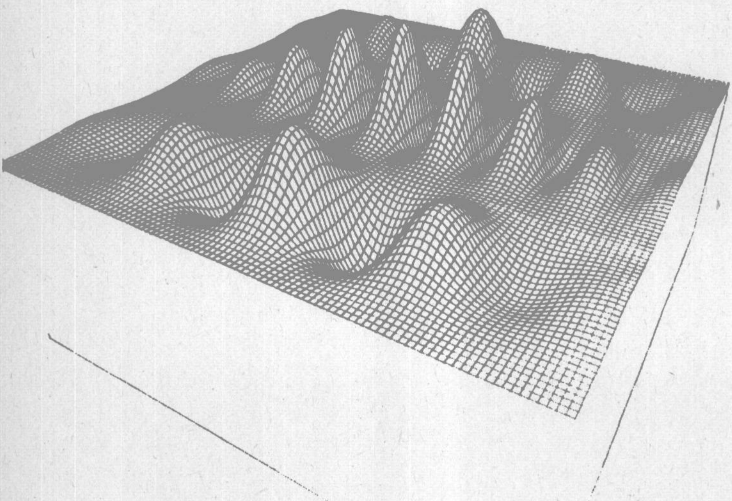
$$\Delta v = \frac{c}{2\omega T_o}$$

amounts to 112 m.p.h. Supposing that the repeated pulse admits a Gaussian envelope, the radar ambiguity surface obtains by periodic superposition of plane wave fronts of the following form:



The figures displayed below in different perspectives provide some impression of the whole radar ambiguity surface for a suitable choice of the parameters T and T_o. The taller the sidelobes, the greater the probability of errors in the measurements, or "ambiguities" in signal epoch and Doppler frequency shift. The narrower (sharper) the width of the radar ambiguity function in a given direction, the higher the accuracy and resolution of the range and range rate measurements if all the other conditions are unchanged. It is desirable, therefore, for the central peak of the radar ambiguity function (cf. Theorem 1 infra) to be narrow, and for there to be as few and as low sidelobes as possible.





From the previous discussion it becomes apparent that it is important to investigate the properties of the radar ambiguity function H(f;.,.). In particular, it is important for radar synthesis to know to what extent the radar ambiguity surface determines the envelope f of the signal involved.

In order to solve this problem, it is helpful to realize the analogies between radar measurements and quantum mechanics. In the phase-space formulation of quantum mechanics the Wigner quasiprobability distribution function associated with the wave function $f \in \mathcal{S}(\mathbf{R}^n)$ is given by

$$P: \mathbf{R}^n \oplus \mathbf{R}^n \ni (q,p) \rightsquigarrow \int_{\mathbf{R}^n} f(q+{}^1/_2 t)\bar{f}(q-{}^1/_2 t)e^{-2\pi i\langle p|t\rangle}dt.$$

Since P gives the probability that the coordinates and the momenta have the values q and p, respectively, it allows an elegant mathematical formulation of the uncertainty relation (cf. Wigner [5]). Moreover, P plays an important rôle in certain statistical reasonings of nuclear physics as well. In the case n=1, the Fourier cotransform of P coincides with the radar ambiguity function H(f;.,.); see Pool [1]. On the other hand, the real Heisenberg nilpotent group $\tilde{A}(\mathbf{R}^n)$ forms the group-theoretic embodiment of the Heisenberg canonical commutation relations. Therefore, nilpotent harmonic analysis seems to be a useful tool also for a treatment of the problems of radar synthesis.

The real Heisenberg nilpotent group $\tilde{A}(\mathbf{R}^n)$ is formed by the unipotent matrices

$$\begin{bmatrix} 1 & x_1 & \dots & x_n & z \\ & 1 & & \bigcirc & y_1 \\ & & \ddots & & \vdots \\ & \bigcirc & & 1 & y_n \\ & & & & 1 \end{bmatrix} = \left(\binom{x}{y}, z\right) = (v,z) \in \mathbf{R}^{2n} \oplus \mathbf{R}$$

with real elements [4]. Observe that $\tilde{A}(\mathbf{R}^n)$ is a connected, simply connected, nilpotent Lie group of step two, i.e., the commutator of any two of its elements belongs to its center Z. It is easily seen that Z may be characterized by $x_1=\dots=x_n=0=y_1=\dots=y_n$, i.e., Z is isomorphic to the additive group $\mathbf{R}$. Moreover, it is not difficult to verify that $\widetilde{A}(\mathbf{R}^n)$ is isomorphic to $\mathbf{R}^{2n} \oplus \mathbf{R}$ with respect to the multiplication rule

$$(v_1,z_1).(v_2,z_2) = (v_1+v_2, z_1+z_2+\tfrac{1}{2}B(v_1,v_2)).$$

Here B denotes the symplectic form

$$B: \mathbf{R}^n \oplus \mathbf{R}^n \ni (v_1,v_2) \rightsquigarrow \langle x_1|y_2\rangle - \langle x_2|y_1\rangle.$$

In particular, the real symplectic group $Sp(n,\mathbf{R})$ forms a subgroup of the automorphism group of $\widetilde{A}(\mathbf{R}^n)$.

Let us recall from Kirillov theory that for all $\lambda \in \mathbf{R}, \lambda \neq 0$, the map U_λ of $\widetilde{A}(\mathbf{R}^n)$ defined by

$$U_\lambda(\binom{x}{y},z)f(t)=e^{2\pi i\lambda(z+\langle t|y\rangle+{}^1/_2\langle x|y\rangle)}f(t+x)$$

with $t \in \mathbf{R}^n$, $f \in \mathcal{S}(\mathbf{R}^n)$, is an unitary continuous irreducible linear representation of $\widetilde{A}(\mathbf{R}^n)$ in the complex Hilbert space $L^2(\mathbf{R}^n)$ which admits the Schwartz-Bruhat space $\mathcal{S}(\mathbf{R}^n)$ as its space of smooth vectors. Furthermore, every unitary irreducible representation (of dimension >1) of $\widetilde{A}(\mathbf{R}^n)$ is unitarily isomorphic to U_λ, for some specific value $\lambda \in \mathbf{R}$ with $\lambda \neq 0$. In the case $\lambda = 1$, we obtain the Schrödinger representation $U = U_1$ of $\widetilde{A}(\mathbf{R}^n)$ as a prototype of an infinite dimensional unitary continuous irreducible linear representation of $\widetilde{A}(\mathbf{R}^n)$. Let us denote its coefficient function by c_U. Then we have by standard results of nilpotent harmonic analysis (cf. [2]):

<u>Theorem 1.</u> For any envelope $f \in \mathcal{S}(\mathbf{R}^n)$ the radar ambiguity function with respect to f satisfies the identity

$$H(f;x,y) = c_U(f;\binom{x}{y},0)$$

for all arrival times $x \in \mathbf{R}^n$ and frequency shifts $y \in \mathbf{R}^n$. In particular, the mapping $(x,y) \rightsquigarrow H(f;x,y)$ is a mod Z square-integrable function of positive type on the real Heisenberg nilpotent group $\tilde{A}(\mathbf{R}^n)$.

Let us mention two consequences of Theorem 1 supra which follow from well-known properties of functions of positive type.

<u>Corollary.</u> The radar ambiguity function $H(f;.,.)$ satisfies

(i) $$\sup_{(x,y) \in \mathbf{R}^n \oplus \mathbf{R}^n} |H(f;x,y)| = H(f;0,0) = \|f\|^2 = 1$$

and

(ii) $$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |H(f;x,y)|^2 dxdy = \|f\|^2 = 1$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$ having L^2-norm $\|f\| = 1$.

By property (i) the radar ambiguity function $H(f;.,.)$ takes on its peak value at the origin, i.e., the ambiguity surface is peaked at the point $(0,0) \in \mathbf{R}^n \oplus \mathbf{R}^n$. From property (ii) a geometric statement of the radar uncertainty principle can be deduced: The volume of the "ambiguity body" is an invariant that does not depend on the specific signal form. The overall uncertainty of the range and range rate measurements is preserved; by changing the signal form, it is possible to change the accuracy of the range and range rate measurement merely in such a manner that a gain for one of the parameters leads to a loss for the other one.

Theorem 1 supra implies an important symmetry property of the radar ambiguity function H. Indeed, the mapping

$$\tau : \left(\begin{pmatrix} x \\ y \end{pmatrix}, z\right) \rightsquigarrow \left(\begin{pmatrix} y \\ -x \end{pmatrix}, z\right)$$

defines an automorphism of $\tilde{A}(\mathbf{R}^n)$ which leaves the center Z pointwise fixed. Therefore, by the Stone-von Neumann-Segal theorem, U and $U^{\tau} = U \circ \tau$ are isomorphic mod Z square-integrable irreducible unitary linear representations of $\tilde{A}(\mathbf{R}^n)$. It is easy to verify that the Fourier cotransform $\overline{\mathcal{F}}_{\mathbf{R}^n}$ defines a unitary isomorphy of U onto U^{τ}, i.e., that the intertwining identity

$$\overline{\mathcal{F}}_{\mathbf{R}^n} \circ U = U^{\tau} \circ \overline{\mathcal{F}}_{\mathbf{R}^n}$$

holds. Thus we have established

<u>Theorem 2.</u> Let $f \in \mathcal{S}(\mathbf{R}^n)$ have L^2-norm $\|f\| = 1$. Then the identity

$$H(f;x,y) = H(\overline{\mathcal{F}}_{\mathbf{R}^n} f; y,-x)$$

holds for all pairs $(x,y) \in \mathbf{R}^n \oplus \mathbf{R}^n$.

It is quite natural to ask whether the radar ambiguity function $H(f;.,.)$ determines uniquely the envelope $f \in \mathcal{S}(\mathbf{R}^n)$ of L^2-norm $\|f\| = 1$. However, the following more general problem reveals itself to be more important in radar synthesis: Given the radar ambiguity surface $H(f;\mathbf{R}^n,\mathbf{R}^n)$ of a standardized function $f \in \mathcal{S}(\mathbf{R}^n)$, determine its linear energy preserving invariants, i.e., the energy preserving linear mappings that transform $H(f;\mathbf{R}^n,\mathbf{R}^n)$ onto itself. The following theorem solves this problem and includes the uniqueness result as a special result.

Theorem 3. Let the functions $f \in \mathcal{S}(\mathbf{R}^n)$ and $f' \in \mathcal{S}(\mathbf{R}^n)$ be given. Suppose $\|f\| = \|f'\| = 1$ and that there exists for any pair of vectors $(x,y) \in \mathbf{R}^n \oplus \mathbf{R}^n$ a pair $(x',y') \in \mathbf{R}^n \oplus \mathbf{R}^n$ such that

$$H(f;x,y) = H(f';x',y').$$

Then there are a unitary operator T of $L^2(\mathbf{R}^n)$ which is unique up to a multiple by the scalar operator $\zeta\, \mathrm{id}_{L^2(\mathbf{R}^n)}$, where $\zeta \in \mathbf{T}$, and a (unique) symplectic linear mapping $u \in Sp(n,\mathbf{R})$ such that the identities

$$f = T(f'),\ (x,y) = u(x',y')$$

hold.

Proof. Let $\mathbf{U}(L^2(\mathbf{R}^n))$ denote the unitary group of the complex Hilbert space $L^2(\mathbf{R}^n)$ equipped with the strong topology. In view of Theorem 1 supra we have to determine the closed subgroup M of $\mathbf{U}(L^2(\mathbf{R}^n))$ consisting of all elements $T \in \mathbf{U}(L^2(\mathbf{R}^n))$ such that for each vector $v = \binom{x}{y} \in \mathbf{R}^n \oplus \mathbf{R}^n$ there exists a vector $v' = \binom{x'}{y'} \in \mathbf{R}^n \oplus \mathbf{R}^n$ so that the identity

$$T \circ U(v,z) \circ T^{-1} = U(v',z)$$

holds for all $z \in \mathbf{R}$. We know that $\overline{\mathcal{F}}_{\mathbf{R}^n} \in M$. Obviously $v' \in \mathbf{R}^{2n}$ depends on $v \in \mathbf{R}^{2n}$ and on the element $T \in M$. Let $v' = u(T)(v)$. Then $u(T): \mathbf{R}^{2n} \longrightarrow \mathbf{R}^{2n}$ defines a homeomorphism. Recall that the symplectic group $Sp(n,\mathbf{R})$ of the phase space $\mathbf{R}^n \oplus \mathbf{R}^n$ forms a subgroup of the automorphism group of $\tilde{A}(\mathbf{R}^n)$ and that the metaplectic group $Mp(n,\mathbf{R})$ is a double covering group of $Sp(n,\mathbf{R})$. It is easily checked that $u(T) \in Sp(n,\mathbf{R})$ for all $T \in M$. Moreover, $u: M \longrightarrow Sp(n,\mathbf{R})$ is a Lie group homomorphism which extends the covering homomorphism j in the short exact sequence

$$\{0\} \longrightarrow (\mathbf{Z}/2\mathbf{Z})\mathrm{id}_{L^2(\mathbf{R}^n)} \longrightarrow Mp(n,\mathbf{R}) \xrightarrow{j} Sp(n,\mathbf{R}) \longrightarrow \{1\}.$$

Since the corresponding short exact sequence for u admits the form

$$\{1\} \longrightarrow \mathbf{T}\mathrm{id}_{L^2(\mathbf{R}^n)} \longrightarrow M \xrightarrow{u} Sp(n,\mathbf{R}) \longrightarrow \{1\},$$

the result follows.-

The unitary operator T can be computed explicitly by means of polarizations. In particular, the preceding theorem reveals the symplectic invariance of the radar ambiguity surface. - Let us now consider the case n=1 in some more detail. Obviously

$$Sp(1,\mathbf{R}) = SL(2,\mathbf{R})$$

and we obtain (cf. [3]):

Theorem 4. The radar ambiguity surface $H(f;\mathbf{R},\mathbf{R})$ with respect to the envelope $f \in \mathcal{S}(\mathbf{R})$ with norm $\|f\| = 1$ is $SO(2,\mathbf{R})$-invariant if and only if there exists a phase factor, i.e., a number $\zeta \in \mathbf{C}$ of modulus $|\zeta| = 1$, and a Hermite-Weber function W_m of degree $m \geq 0$ such that

$$f = \zeta W_m$$

holds.

The figures displayed below show the radar ambiguity surface with respect to the envelope $f = W_m$ (which can be expressed in terms of the Laguerre-Weber function of degree m) in the case $m = 1,\dots,4$.