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FOURIER TRANSFORMS IN THE COMPLEX DOMAIN

BY

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AND

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Dedicated by the surviving author

to

PROFESSORS G. H. HARDY *and* J. E. LITTLEWOOD

the teachers of us both

PREFACE

The present book represents a definitive statement of the results obtained by the late Mr. R. E. A. C. Paley and myself during Mr. Paley's year as Rockefeller Fellow at the Massachusetts Institute of Technology (1932–1933). Mr. Paley was killed on April 7 in a skiing accident in the Canadian Rockies, during a short vacation which he had taken from our joint work. I have written elsewhere of the great loss to mathematics by his death; here let me only state the condition in which our joint work was left. Our method of collaboration had been most informal. We had worked together with a blackboard before us, and when we had covered it with our joint comments, one or the other would copy down what was relevant, and reduce it to a preliminary written form. Most of our work went through several versions, in writing which both authors took part. Even in that part of the research committed to writing since Mr. Paley's death, it is completely impossible to determine how much is new and how much is a reminiscence of our many conversations.

A part of our work was published in the form of a series of notes in the Transactions of the American Mathematical Society. This work covered a great variety of topics, but was unified by the central idea of the application of the Fourier transform in the complex domain. I had long been convinced of the importance of the Fourier-Mellin transforms as a tool in analysis. Their introduction is of course no novelty, but I know of no systematic development of their methodical use. Perhaps the nearest approach to such a development is to be found in the researches of H. Bohr, Jessen and Besicovitch on almost periodic functions in the complex domain. However, nobody seems to have realized anything like the scope of the method. With its aid, we were able to attack such diverse analytic questions as those of quasi-analytic functions, of Mercer's theorem on summability, of Milne's integral equation of radiative equilibrium, of the theorems of Münz and Szász concerning the closure of sets of powers of an argument, of Titchmarsh's theory of entire functions of semi-exponential type with real negative zeros, of trigonometric interpolation and developments in polynomials of the form

$$\sum_{1}^{N} A_n e^{i\lambda_n x},$$

of lacunary series, of generalized harmonic analysis in the complex domain, of the zeros of random functions, and many others. We came to believe that an analytic method of such scope is entitled to an independent treatise.

The American Mathematical Society has done me the honor of requesting me to deliver the Williamstown Colloquium Lectures for 1934. While such lectures have not previously been an account of collaborative work, my best available work has been collaborative, and I have offered it for the lectures in question.

I wish to thank the American Mathematical Society for its invitation, and for its acceptance of our plans. I wish to thank my students, Messrs S. S. Saslaw, H. Malin, and N. Levinson, for most valuable and painstaking work of revision, compilation, and criticism. Mr. Levinson, in particular, has added much to the content of Chapter I. I wish to thank my colleague, Professor Eberhard Hopf, for permission to incorporate into this book the material of §17, which was our joint work. Furthermore, in my own name and in the name of my dead co-author, I wish to thank Professor J. D. Tamarkin of Brown University for his untiring encouragement, advice, and criticism, without which this book would not have come into existence.

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CAMBRIDGE, MASSACHUSETTS, MARCH 1, 1934.

CONTENTS

Introduction

1. **Plancherel's Theorem.** In a book such as the present, unified rather by the repeated use of a number of methods than by a great homogeneity of content, it is quite necessary to start with a brief account of the background of scientific knowledge presupposed and a tabulation of the principal methods. The background of knowledge presupposed in the greater part of this treatise is roughly that covered in Titchmarsh's very useful *Theory of Functions*. The tools most used are the following:

(1) Integration by parts, and other similar inversions of the order of an absolutely convergent double integral;

(2) The "mutilation" of the function: that is, the replacement of a function by a function identical with the first over a finite range, and vanishing outside that range;

(3) The Schwarz inequality

$$\left[\int_a^b |f(x)g(x)|\,dx\right]^2 \leqq \int_a^b |f(x)|^2\,dx \int_a^b |g(x)|^2\,dx, \tag{1.01}$$

and similar inequalities for sums, series, etc.;

(4) The Weyl form of the Riesz-Fischer theorem, to the effect that if a sequence of functions $\{f_n(x)\}$ of L_2 converges in the mean in the sense

$$\lim_{m,n\to\infty} \int_a^b |f_m(x) - f_n(x)|^2\,dx = 0, \tag{1.02}$$

then there exists a function $f(x)$ of L_2 to which the sequence converges in the mean in the sense

$$\lim_{m\to\infty} \int_a^b |f_m(x) - f(x)|^2\,dx = 0. \tag{1.03}$$

(5) The theorem that if a sequence of functions converges in the mean to one limit, and converges in the ordinary sense to another, then these two limits differ at most over a null set;

(6) Methods of summability and averaging, in particular theorems of Abelian and Tauberian types:*

(7) Methods depending on the Plancherel and the Parseval theorems concerning Fourier transforms.†

* Cf. Wiener, *The Fourier Integral and Certain of its Applications*, Cambridge, 1933. The Tauberian theorems of this book are not to be found in the book of Titchmarsh.

† A good elementary study of such questions is to be found in S. Bochner's *Vorlesungen über Fouriersche Integrale*, Leipzig, 1932. The treatment in Wiener's book (see above) is slightly more advanced.

Throughout this book we shall assume on the part of the reader familiarity with the theory and use of the Lebesgue integral and with the appropriate notations. In particular, we shall make repeated use of the notation L to represent the class of measurable, absolutely integrable functions, and of the notation L_p for the class of those measurable functions, the pth power of whose modulus is integrable. However, we shall have little to do with other classes than L and L_2.

In the theory of the Fourier integral, the fundamental theorem for the class L_2 is that of Plancherel. It reads as follows:

PLANCHEREL'S THEOREM. *Let $f(x)$ belong to L_2 over $(-\infty, \infty)$. Then there exists a function $g(u)$ belonging to L_2 over $(-\infty, \infty)$, and such that*

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} \left| g(u) - (2\pi)^{-1/2} \int_{-A}^{A} f(x)\, e^{iux}\, dx \right|^2 du = 0. \tag{1.04}$$

Furthermore

$$\int_{-\infty}^{\infty} | g(u) |^2 \, du = \int_{-\infty}^{\infty} | f(x) |^2 \, dx, \tag{1.05}$$

and

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} \left| f(x) - (2\pi)^{-1/2} \int_{-A}^{A} g(u)\, e^{-iux}\, du \right|^2 dx = 0. \tag{1.06}$$

The function $g(u)$ is called the Fourier Transform of $f(x)$. It is determined except over a set of points of zero measure.

In case

$$h(u) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx \tag{1.07}$$

exists, we have $g(u) = h(u)$ almost everywhere.

An important corollary of Plancherel's theorem is

PARSEVAL'S THEOREM. *Let $f_1(x)$ and $f_2(x)$ both belong to L_2, and let them have the Fourier transforms $g_1(u)$ and $g_2(u)$, respectively. Then*

$$\int_{-\infty}^{\infty} g_1(u)\, g_2(u)\, du = \int_{-\infty}^{\infty} f_1(x)\, f_2(-x)\, dx. \tag{1.08}$$

In particular,

$$\int_{-\infty}^{\infty} g_1(u)\, g_2(u)\, e^{-iux}\, du = \int_{-\infty}^{\infty} f_1(y)\, f_2(x - y)\, dy. \tag{1.09}$$

Thus, if $g_1(u)\, g_2(u)$ belongs to L_2 as well as both its factors, it is the Fourier transform of

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} f_1(y)\, f_2(x - y)\, dy. \tag{1.10}$$

This will also be true whenever $f_1(x)$, $f_2(x)$, and (1.10) all belong to L_2.

2. The Fourier transform of a function vanishing exponentially. Let us suppose that $f(x)$ is measurable, and of summable square over any finite interval. Let

$$f(x) = \begin{cases} O(e^{-\mu x}) & [x \to \infty]; \\ O(e^{\lambda x}) & [x \to -\infty]. \end{cases} \tag{2.1}$$

In case $-\lambda < \sigma < \mu$, the Fourier transform of $f(x)\, e^{\sigma x}$ will then be

$$F(\sigma, t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x)\, e^{(\sigma + it)x}\, dx. \tag{2.2}$$

However, this converges absolutely and uniformly over every range $-\lambda + \epsilon < \sigma < \mu - \epsilon$. Thus by a well known theorem from the theory of functions of a complex variable,

$$F(\sigma + it) = F(\sigma, t) \tag{2.3}$$

will be an analytic function of $\sigma + it$ over the interior of the strip $-\lambda < \sigma < \mu$. Furthermore, over any strip $-\lambda + \epsilon < \sigma < \mu - \epsilon$ we shall have

$$\begin{aligned} \int_{-\infty}^{\infty} | F(\sigma, t) |^2\, dt &= \int_{-\infty}^{\infty} | f(x) |^2\, e^{2\sigma x}\, dx \\ &< \text{const.} \int_0^{\infty} e^{-2\epsilon x}\, dx + \text{const.} \int_{-\infty}^{0} e^{2\epsilon x}\, dx = \text{const.} \end{aligned} \tag{2.4}$$

We have thus proved

THEOREM I. *If $f(x)$ is measurable, of summable square over every finite interval, and a function satisfying* (2.1), *for* $-\lambda < \mu$, *then* (2.2) *defines a function* $F(\sigma + it)$ *analytic over the interior of the strip* $-\lambda < \sigma < \mu$; *and over any interior strip* $-\lambda + \epsilon \leqq \sigma \leqq \mu - \epsilon$, $\int_{-\infty}^{\infty} | F(\sigma + it) |^2\, dt$ *is bounded.*

3. The Fourier transform of a function in a strip. Let $F(\sigma + it)$ be a function of the complex variable $s = \sigma + it$, which is analytic in and on the boundary of the strip $-\lambda \leqq \sigma \leqq \mu$, and let

$$\int_{-\infty}^{\infty} | F(\sigma + it) |^2\, dt < \text{const.} \qquad [-\lambda \leqq \sigma \leqq \mu]. \tag{3.01}$$

Then by Cauchy's theorem, if A is large enough and $-\lambda < \sigma < \mu$,

$$F(s) = \frac{1}{2\pi i} \left[\int_{-\lambda + Ai}^{-\lambda - Ai} + \int_{-\lambda - Ai}^{\mu - Ai} + \int_{\mu - Ai}^{\mu + Ai} + \int_{\mu + Ai}^{-\lambda + Ai} \right] \frac{F(z)}{z - s}\, dz. \tag{3.02}$$

By a further integration, if B is large enough,

$$F(s) = \frac{1}{2\pi i} \int_B^{B+1} dA \left[\int_{-\lambda + Ai}^{-\lambda - Ai} + \int_{-\lambda - Ai}^{\mu - Ai} + \int_{\mu - Ai}^{\mu + Ai} + \int_{\mu + Ai}^{-\lambda + Ai} \right] \frac{F(z)}{z - s}\, dz. \tag{3.03}$$

Now,

$$\left|\frac{1}{2\pi i}\int_B^{B+1} dA \int_{\mu+Ai}^{-\lambda+Ai} \frac{F(z)}{z-s}\,dz\right| = \frac{1}{2\pi}\left|\int_{-\lambda}^{\mu} dz \int_B^{B+1} \frac{F(z+Ai)}{z+Ai-s}\,dA\right|$$

(3.04)
$$\leqq \frac{1}{2\pi}\int_{-\lambda}^{\mu} dz \left\{\int_B^{B+1} |F(z+Ai)|^2\,dA \int_B^{B+1} \frac{dA}{|z+Ai-s|^2}\right\}^{1/2}$$

$$\leqq \text{const.} \int_{-\lambda}^{\mu} d\sigma \left\{\int_B^{B+1} |F(\sigma+it)|^2\,dt\right\}^{1/2}.$$

By (3.01) it follows that

(3.05)
$$\begin{cases} \left\{\int_B^{B+1} |F(\sigma+it)|^2\,dt\right\}^{1/2} < \text{const.}, \\ \lim_{B\to\infty} \left\{\int_B^{B+1} |F(\sigma+it)|^2\,dt\right\}^{1/2} = 0. \end{cases}$$

It is a familiar theorem in the theory of the Lebesgue integral, that if a sequence of integrable functions converges boundedly to a limit, and the integral of that limit exists, then the limit of the integral of a function of the sequence is the integral of the limit. Thus

(3.06)
$$\lim_{B\to\infty} \frac{1}{2\pi i}\int_B^{B+1} dA \int_{\mu+Ai}^{-\lambda+Ai} \frac{F(z)}{z-s}\,dz = 0.$$

Similarly,

(3.07)
$$\lim_{B\to\infty} \frac{1}{2\pi i}\int_B^{B+1} dA \int_{-\lambda-Ai}^{\mu-Ai} \frac{F(z)}{z-s}\,dz = 0.$$

Thus

(3.08)
$$\begin{aligned} F(s) &= \lim_{B\to\infty} \frac{1}{2\pi}\int_B^{B+1} dA \int_{-A}^{A} \left\{\frac{F(\mu+iy)}{\mu+iy-s} - \frac{F(-\lambda+iy)}{-\lambda+iy-s}\right\} dy \\ &= \lim_{B\to\infty} \left\{\frac{1}{2\pi}\int_{-B}^{B} \frac{F(\mu+iy)}{\mu+iy-s}\,dy - \frac{1}{2\pi}\int_{-B}^{B} \frac{F(-\lambda+iy)}{-\lambda+iy-s}\,dy \right. \\ &\quad + \frac{1}{2\pi}\int_B^{B+1} \frac{(B+1-y)}{\mu+iy-s} F(\mu+iy)\,dy \\ &\quad - \frac{1}{2\pi}\int_B^{B+1} \frac{(B+1-y)\,F(-\lambda+iy)}{-\lambda+iy-s}\,dy \\ &\quad + \frac{1}{2\pi}\int_{-B-1}^{-B} \frac{(B+1+y)\,F(\mu+iy)}{\mu+iy-s}\,dy \\ &\quad \left. - \frac{1}{2\pi}\int_{-B-1}^{-B} \frac{(B+1+y)\,F(-\lambda+iy)}{-\lambda+iy-s}\,dy\right\}. \end{aligned}$$

Now,

$$\left|\frac{1}{2\pi}\int_{B}^{B+1}\frac{(B+1-y)\,F(\mu+iy)}{\mu+iy-s}\,dy\right| \leqq \frac{1}{2\pi}\left\{\int_{B}^{B+1}\left|\frac{B+1-y}{\mu+iy-s}\right|^{2}dy\right.$$

$$\left.\times\int_{B}^{B+1}|F(\mu+iy)|^{2}\,dy\right\}^{1/2} \tag{3.09}$$

$$\leqq \text{const.}\left\{\int_{B}^{B+1}|F(\mu+iy)|^{2}\,dy\right\}^{1/2}$$

It follows at once from (3.01) that

$$\lim_{B\to\infty}\frac{1}{2\pi}\int_{B}^{B+1}\frac{(B+1-y)\,F(\mu+iy)}{(\mu+iy-s)}\,dy = 0. \tag{3.10}$$

A similar argument will enable us to eliminate three more terms from (3.08), and we get

$$F(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\mu+iy)}{\mu+iy-s}\,dy - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(-\lambda+iy)}{-\lambda+iy-s}\,dy. \tag{3.11}$$

This is a sufficiently important result to be numbered as a theorem. We have

THEOREM II. *Let $F(s)$ be analytic over $-\lambda \leqq \sigma \leqq \mu$, and let* (3.01) *hold over this region. Then if s is interior to this region,* (3.11) *is valid.*

The use of the Schwarz inequality gives us

$$\left|\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{F(\mu+iy)}{(\mu+iy-s)}\,dy\right| \leqq \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}|F(\mu+iy)|^{2}\,dy\int_{-\infty}^{\infty}\frac{dy}{|\mu+iy-s|^{2}}\right\}^{1/2}, \tag{3.12}$$

or

THEOREM III. *Under the hypothesis of Theorem II, $F(s)$ is bounded over any region $-\lambda+\epsilon \leqq \sigma \leqq \mu-\epsilon$.*

Let us put

$$f(\sigma, x) = (2\pi)^{-1/2}\,\text{l.i.m.}\int_{-A}^{A}F(\sigma+it)\,e^{-itx}\,dt. \tag{3.13}$$

Let us also put

$$\phi(x) = 0\ [x<0]; \qquad \phi(x) = e^{-\alpha x}\ [x>0]; \qquad \alpha > 0. \tag{3.14}$$

We shall have

$$\int_{-\infty}^{\infty}\phi(x)\,e^{ixy}\,dx = \int_{0}^{\infty}e^{ixy-\alpha x}\,dx = \frac{1}{\alpha-iy}. \tag{3.15}$$

Plancherel's theorem yields

$$\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-A}^{A}\frac{e^{-ixy}}{\alpha-iy}\,dy = \begin{cases}0 & [x<0]\\ e^{-\alpha x} & [x>0]\end{cases}\ \text{if } \alpha > 0. \tag{3.16}$$

Similarly, if $\alpha < 0$,

$$\underset{A\to\infty}{\text{l. i. m.}} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{ixy}}{\alpha - iy} dy = \begin{cases} -e^{-\alpha x} & [x < 0] \\ 0 & [x > 0]. \end{cases} \tag{3.17}$$

Thus by the Parseval theorem, since $-\lambda + \epsilon < \sigma < \mu - \epsilon$,

$$\int_{-\infty}^{0} f(-\lambda, x)\, e^{(\sigma+\lambda)x}\, e^{itx}\, dx = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{F(-\lambda + iy)}{[-\lambda - \sigma - (t-y)i]} dy. \tag{3.18}$$

In exactly the same way,

$$\int_{0}^{\infty} f(\mu, x)\, e^{(\sigma-\mu)x}\, e^{itx}\, dx = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{F(\mu + iy)}{[\mu - \sigma - (t-y)i]} dy. \tag{3.19}$$

Thus by (3.11)

$$F(s) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} f(-\lambda, x)\, e^{(\sigma+\lambda)x}\, e^{itx}\, dx + \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} f(\mu, x)\, e^{(\sigma-\mu)x}\, e^{itx}\, dx. \tag{3.20}$$

Another use of Plancherel's theorem gives us

$$f(\sigma, x) = \begin{cases} f(-\lambda, x)\, e^{(\sigma+\lambda)x} & [-\infty < x < 0]; \\ f(\mu, x)\, e^{(\sigma-\mu)x} & [0 < x < \infty]; \end{cases} \tag{3.21}$$

and consequently if for a particular α

$$f(x) = f(\alpha, x) e^{-\alpha x} \qquad [-\lambda < \alpha < \mu], \tag{3.22}$$

we get

$$f(\sigma, x) = f(x) e^{\sigma x} \qquad [-\lambda < \sigma < \mu]. \tag{3.23}$$

An immediate result is that

$$\lim_{\sigma, \sigma_1 \to \mu - 0} \int_{A}^{B} | f(\sigma, x) - f(\sigma_1, x) |^2\, dx = 0. \tag{3.24}$$

Moreover, if B is positive and A negative,

$$\int_{B}^{\infty} | f(\sigma, x) |^2\, dx \leqq \int_{B}^{\infty} | f(\mu, x) |^2\, dx, \tag{3.25}$$

and

$$\int_{-\infty}^{A} | f(\sigma, x) |^2\, dx \leqq \int_{-\infty}^{A} | f(-\lambda, x) |^2\, dx. \tag{3.26}$$

As a consequence,

$$\lim_{\sigma, \sigma_1 \to \mu - 0} \int_{-\infty}^{\infty} | f(\sigma, x) - f(\sigma_1, x) |^2\, dx = 0. \tag{3.27}$$

Thus by Weyl's lemma to the Riesz-Fischer theorem, there exists in L_2 a function $f_1(x)$ such that

$$\lim_{\sigma \to \mu - 0} \int_{-\infty}^{\infty} | f(\sigma, x) - f_1(x) |^2 \, dx = 0. \tag{3.28}$$

By a Fourier transformation and the use of Plancherel's theorem, there exists a function $F_1(t)$ such that

$$\lim_{\sigma \to \mu - 0} \int_{-\infty}^{\infty} | F(\sigma + it) - F_1(t) |^2 \, dt = 0. \tag{3.29}$$

However, we already know that

$$\lim_{\sigma \to \mu - 0} F(\sigma + it) = F(\mu + it), \tag{3.30}$$

and hence,

$$\lim_{\sigma \to \mu - 0} \int_{-\infty}^{\infty} | F(\sigma + it) - F(\mu + it) |^2 \, dt = 0. \tag{3.31}$$

By a further Fourier transformation,

$$\lim_{\sigma \to \mu - 0} \int_{-\infty}^{\infty} | f(\sigma, x) - f(\mu, x) |^2 \, dx = 0. \tag{3.32}$$

Similarly,

$$\lim_{\sigma \to -\lambda + 0} \int_{-\infty}^{\infty} | f(\sigma, x) - f(-\lambda, x) |^2 \, dx = 0. \tag{3.33}$$

However,

$$\lim_{\sigma \to \mu} f(\sigma, x) = f(-\lambda, x)\, e^{(\mu+\lambda)x} \qquad [-\infty < x < 0], \tag{3.34}$$

and

$$\lim_{\sigma \to -\lambda} f(\sigma, x) = f(\mu, x)\, e^{-(\mu+\lambda)x} \qquad [0 < x < \infty]. \tag{3.35}$$

Thus with the possible exception of a null set,

$$f(\sigma, x) = f(\mu, x) e^{(\sigma-\mu)x} = f(-\lambda, x) e^{(\sigma+\lambda)x} \qquad [-\infty < x < \infty]. \tag{3.36}$$

This yields us

THEOREM IV. *Under the hypotheses of Theorem* II, *there exists a measurable function* $f(x)$, *such that*

$$\int_{-\infty}^{\infty} | f(x) |^2 e^{2\mu x} \, dx < \infty, \qquad \int_{-\infty}^{\infty} | f(x) |^2 e^{-2\lambda x} \, dx < \infty, \tag{3.37}$$

and that over the closed interval $-\lambda \leqq \sigma \leqq \mu$,

$$F(\sigma + it) = \underset{A \to \infty}{\text{l.i.m.}}\ (2\pi)^{-1/2} \int_{-A}^{A} f(x)\, e^{x(\sigma+it)} \, dx. \tag{3.38}$$

It follows from Theorems I and IV that the extreme boundaries of the interval over which $F(\sigma + it)$ belongs uniformly to L_2 as a function of t are given by the boundaries of convergence of the integral

$$\int_{-\infty}^{\infty} |f(x)|^2 e^{2\sigma x}\, dx.$$

4. The Fourier transform of a function in a half-plane. In particular, $F(\sigma + it)$ will belong to L_2 in every ordinate of a right half-plane when and only when

$$\int_{-\infty}^{\infty} |f(x)|^2 e^{2\sigma x}\, dx < \infty \tag{4.01}$$

for all sufficiently large σ, and will belong to L_2 *uniformly* in such a half-plane when and only when

$$\lim_{\sigma \to \infty} \int_{-\infty}^{\infty} |f(x)|^2 e^{2\sigma x}\, dx < \infty. \tag{4.02}$$

This latter contingency can only occur when $f(x)$ vanishes almost everywhere for positive values of its argument. Otherwise there will be some interval (a, b) $[b > a > 0]$ over which

$$\int_{a}^{b} |f(x)|^2\, dx = I > 0. \tag{4.03}$$

We shall then have

$$\int_{-\infty}^{\infty} |f(x)|^2 e^{2\sigma x}\, dx \geqq e^{2\sigma a} I, \tag{4.04}$$

which is contrary to our assumption.

Conversely, if $f(x)$ vanishes for positive values of its argument, and if, for some λ,

$$\int_{-\infty}^{0} |f(x)|^2 e^{-2\lambda x}\, dx < \infty, \tag{4.05}$$

the function $F(\sigma + it)$ defined by (3.38) will belong uniformly to L_2 as a function of t for $\sigma \geqq -\lambda$. In particular, we have

THEOREM V. *The two following classes of analytic functions are identical:* (1) *the class of all functions $F(\sigma + it)$ analytic for $\sigma > 0$, and such that*

$$\int_{-\infty}^{\infty} |F(\sigma + it)|^2\, dt < \text{const.} \qquad [0 < \sigma < \infty]; \tag{4.06}$$

(2) *the class of all functions defined by*

$$F(\sigma + it) = \underset{A \to \infty}{\text{l.i.m.}}\, (2\pi)^{-1/2} \int_{-A}^{0} f(x)\, e^{x(\sigma + it)}\, dx, \tag{4.07}$$

where $f(x)$ belongs to L_2 over $(-\infty, 0)$.

We shall have

$$\text{(4.08)} \qquad \underset{\sigma\to 0}{\text{l.i.m.}}\, F(\sigma + it) = \underset{A\to\infty}{\text{l.i.m.}}\, (2\pi)^{-1/2} \int_{-A}^{0} f(x)\, e^{itx}\, dx.$$

5. Theorems of the Phragmén-Lindelöf type. Let us again consider functions $F(\sigma + it)$, analytic over $-\lambda \leqq \sigma \leqq \mu$. Let us assume that

$$\text{(5.01)} \qquad \int_{-\infty}^{\infty} |F(-\lambda + it)|^2\, dt < \infty, \qquad \int_{-\infty}^{\infty} |F(\mu + it)|^2\, dt < \infty,$$

and that

$$\text{(5.02)} \qquad |F(\sigma + it)| < M \qquad [-\lambda \leqq \sigma \leqq \mu].$$

In place of (3.04) we shall have for sufficiently large B

$$\text{(5.03)} \qquad \begin{aligned} \left| \frac{1}{2\pi i} \int_{B}^{B+1} dA \int_{\mu+Ai}^{-\lambda+Ai} \frac{F(z)}{z - s}\, dz \right| &\leqq \frac{1}{2\pi} \int_{-\lambda}^{\mu} dz \int_{B}^{B+1} \frac{M\, dA}{|z + A_i - s|} \\ &\leqq \frac{M(\mu + \lambda)}{2\pi} \frac{1}{B - t}, \end{aligned}$$

so that (3.08) is established as before. Thus (3.11) is valid, and the whole argument up to (3.21) is repeated unchanged. The sole difference is that the argument this time is so turned as to *prove* that $F(\sigma + it)$ is the Fourier transform of a function $f(\sigma, x)$ belonging to L_2, instead of initially assuming it, or the equivalent fact that $F(\sigma + it)$ belongs uniformly to L_2.

It results immediately from (3.21) that $f(\sigma, t)$ belongs uniformly to L_2 over $(-\lambda, \mu)$, and hence that $F(\sigma + it)$ belongs uniformly to L_2 over this interval. We have thus proved

THEOREM VI. *If* (5.01) *and* (5.02) *are satisfied, the hypotheses and hence the conclusions of Theorems* II, III, *and* IV *are valid.*

We now appeal to the classical theorem of Phragmén and Lindelöf.* This asserts that if $F(\sigma + it)$ is analytic for $-\lambda \leqq \sigma \leqq \mu$, if

$$\text{(5.04)} \qquad F(\sigma + it) = O(e^{e^{\rho t}}) \qquad [\rho < \mu + \lambda],$$

and if $F(-\lambda + it)$ and $F(\mu + it)$ are bounded, then $F(\sigma + it)$ is bounded for all t and for $-\lambda \leqq \sigma \leqq \mu$. Without any assumption further than that $F(s)$ belongs to L_2 for the ordinates at $-\lambda$ and μ, it will then follow that if $F(s)$ is analytic up to and including these ordinates and in the strip between them, and if (5.04) is fulfilled, the analytic function

$$\text{(5.05)} \qquad F_\epsilon(\sigma + it) = \frac{1}{\epsilon} \int_{t}^{t+\epsilon} F(\sigma + i\tau)\, d\tau = \frac{1}{2\epsilon} \int_{\sigma+it}^{\sigma+it+i\epsilon} F(s)\, ds$$

* E. C. Titchmarsh, *Theory of Functions*, p. 178 ff.

will be bounded and will satisfy (5.04), and hence by the argument with which we have proved Theorem VI,

$$\int_{-\infty}^{\infty} |F_\epsilon(\sigma + it)|^2\, dt \leqq \int_{-\infty}^{\infty} |F_\epsilon(-\lambda + it)|^2\, dx + \int_{-\infty}^{\infty} |F_\epsilon(\mu + it)|^2\, dt \qquad [-\lambda \leqq \sigma \leqq \mu]. \tag{5.06}$$

However, it is a well known theorem that if $\phi(x)$ belongs to L_2,

$$\underset{\epsilon\to 0}{\mathrm{l.i.m.}}\, \frac{1}{\epsilon} \int_x^{x+\epsilon} \phi(\xi)\, d\xi = \phi(x). \tag{5.07}$$

Moreover, we have

$$\begin{aligned}\int_\alpha^\beta |F(\sigma + it)|^2\, dt &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} |F_\epsilon(\sigma + it)|^2\, dt \\ &\leqq \int_{-\infty}^{\infty} |F(-\lambda + it)|^2\, dt + \int_{-\infty}^{\infty} |F(\mu + it)|^2\, dt,\end{aligned} \tag{5.08}$$

and hence

$$\int_{-\infty}^{\infty} |F(\sigma + it)|^2\, dt < \text{const.} \qquad [-\lambda \leqq \sigma \leqq \mu]. \tag{5.09}$$

This yields us

THEOREM VII. *If $F(s)$ is analytic over $-\lambda \leqq \sigma \leqq \mu$, if (5.01) is satisfied, and if (5.04) is satisfied, the conclusions of Theorems II, III and IV are valid.*

We now turn our attention from the strip to the half-plane. Let $F(s)$ be analytic for $\sigma \geqq 0$, let

$$\int_{-\infty}^{\infty} |F(it)|^2\, dt < \infty, \tag{5.10}$$

and let

$$|F(\sigma + it)| \leqq \text{const.} \qquad [0 \leqq \sigma < \infty]. \tag{5.11}$$

As before, we have

$$F(s) = \lim_{B\to\infty} \frac{1}{2\pi} \int_B^{B+1} dA \int_{-A}^{A} \left\{ \frac{F(\mu + iy)}{\mu + iy - s} - \frac{F(iy)}{iy - s} \right\} dy, \tag{5.12}$$

provided μ is large enough. This may be written

$$F(s) = \lim_{B\to\infty} \frac{1}{2\pi} \int_B^{B+1} dA \int_{-A}^{A} \frac{F(\mu + iy)}{\mu + iy - s}\, dy - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(iy)}{iy - s}\, dy \tag{5.13}$$

by the argument of (3.10) and (3.11). Now if μ is large enough, and

$$\Re(s) > 0$$