THE THEORY OF LIE DERIVATIVES AND ITS APPLICATIONS

KENTARO YANO

THE THEORY OF LIE DERIVATIVES AND ITS APPLICATIONS

BY

KENTARO YANO

NORTH-HOLLAND PUBLISHING CO. — AMSTERDAM
P. NOORDHOFF LTD. — GRONINGEN

This book is dedicated to
Prof. Dr. J. A. Schouten
who has been a pioneer in the field of modern differential geometry.

SOLE AGENCY FOR U.S.A.
INTERSCIENCE PUBLISHERS, INC. NEW YORK

PRINTED IN THE NETHERLANDS

THE THEORY OF LIE DERIVATIVES AND ITS APPLICATIONS

BIBLIOTHECA MATHEMATICA

A Series of Monographs on Pure and Applied Mathematics

Volume III

★

Edited with the cooperation of

THE „MATHEMATISCH CENTRUM"

and

THE „WISKUNDIG GENOOTSCHAP"
at Amsterdam

★

Editors

N. G. DE BRUIJN
D. VAN DANTZIG
J. DE GROOT

PREFACE

Since the theory of continuous groups of transformations was inaugurated by S. Lie and F. Engel, the groups of motions in Riemannian spaces were studied by L. Bianchi, G. Fubini, W. Killing, G. Ricci and others.

On the other hand, the idea of spaces with a linear connexion was introduced by E. Cartan, J. A. Schouten and H. Weyl and the affine and projective motions in these spaces were first considered by L. P. Eisenhart and M. S. Knebelman.

In 1931, W. Slebodzinski introduced a new differential operator, later called by D. van Dantzig that of Lie derivation, which can be applied to scalars, vectors, tensors and affine connexions and which proved to be a powerful instrument in the study of groups of automorphisms. Using this operator, D. van Dantzig showed that his n-dimensional projective space described by $n+1$ homogeneous curvilinear coordinates can be regarded as an $(n+1)$-dimensional space with a linear connexion which admits a one-parameter group of affine motions. He applied also the idea of Lie derivation to physics.

Since then the deformations of curves, subspaces and spaces themselves as well as groups of motions, affine motions, projective motions and conformal motions were extensively studied by L. Berwald, E. Cartan, N. Coburn, E. T. Davies, P. Dienes, A. Duschek, L. P. Eisenhart, F. A. Ficken, H. A. Hayden, V. Hlavatý, E. R. van Kampen, M. S. Knebelman, T. Levi-Civita, J. Levine, W. Mayer, A. J. McConnel, A. D. Michal, H. P. Robertson, S. Sasaki, J. A. Schouten, J. L. Synge, A. H. Taub, H. C. Wang, the present author and others.

The Lie derivatives of general geometric objects were studied by A. Nijenhuis, Y. Tashiro and the present author.

It is now a well-known fact that, if an n-dimensional space admits a group of motions, affine motions, projective motions or conformal motions of the maximum order $\frac{1}{2}n(n+1)$, n^2+n, n^2+2n or $\frac{1}{2}(n+1)(n+2)$ respectively, the space is of constant curvature, affinely flat, projectively Euclidean or conformally Euclidean.

In 1947, I. P. Egorov began the study of spaces which have a nonvanishing curvature tensor and which admit a group of automorphisms of the maximum order. Investigations in this direction were carried out by Y. Mutō, G. Vranceanu, H. C. Wang and the present author.

Chapters I—VII of the present book are devoted to the above-mentioned publications.

The automorphisms in Finsler spaces, Cartan spaces, general affine and projective spaces of geodesics and general affine and projective spaces of k-spreads were studied also very extensively by the use of Lie derivatives by R. S. Clark, E. T. Davies, H. Hiramatu, Y. Katsurada, M. S. Knebelman, D. D. Kosambi, B. Laptev, Gy. Soós, B. Su, K. Takano, H. C. Wang, the present author and others. Chapter VIII contains the theory of Lie derivatives and its applications in these spaces.

Chapter IX is devoted to the study of global properties of the groups of motions in a compact orientable Riemannian space. The method used in this Chapter is due to S. Bochner and A. Lichnerowicz.

The last Chapter is devoted to a brief exposition on the almost complex spaces and to some problems which can be dealt with by the use of Lie derivatives.

There is a tendency of developing the theory of Lie derivatives from the point of view of the theory of fibre bundles. But such an investigation has just been started and it seems to the author that it is still premature to give an exposition of the results already obtained. We only refer to the recent papers by R. S. Palais, N. H. Kuiper and the present author.

The bibliography at the end of the book contains only the papers and books quoted in the text and those of which the author may suppose that they are of interest for the readers.

The author wishes to express here his hearty thanks to Prof. J. A. Schouten who read the manuscript and gave many valuable suggestions. The author wishes to thank also the editors of Bibliotheca Mathematica, Prof. D. van Dantzig, Prof. J. de Groot and Prof. N. G. de Bruijn for their most agreeable collaboration.

The author appreciates very much the kind help from his Dutch friends at the Mathematical Centre and the University of Amsterdam. Miss P. Brouwer looked through the manuscript and improved the English of the text. The author's sincere thanks go to all of them.

Amsterdam, April 14, 1955 KENTARO YANO

CONTENTS

CHAPTER I

INTRODUCTION

§ 1. Motions in a Riemannian space.

Consider an n-dimensional Riemannian space V_n [1] of class C^ω [2] covered by a set of neighbourhoods with coordinates ξ^κ and endowed with the fundamental quadratic differential form

$$(1.1) \qquad ds^2 = g_{\lambda\kappa}(\xi)d\xi^\lambda d\xi^\kappa,^{3\ 4}$$

where the Greek indices $\kappa, \lambda, \mu, \nu, \ldots$ run over the range $1, 2, \ldots, n$. We write (κ) to denote the system of coordinates ξ^κ.

In the V_n referred to (κ), we consider a point transformation

$$(1.2) \qquad T\colon\ '\xi^\kappa = f^\kappa(\xi^\nu); \qquad \text{Det}\,(\partial_\lambda f^\kappa) \neq 0$$

of class C^ω [5] which establishes a one-to-one correspondence between the points of a region R and those of some other region $'R$, where ∂_λ stands for the partial derivation $\partial/\partial\xi^\lambda$.

During this point transformation, a point ξ^κ in R is carried to a point $'\xi^\kappa$ in $'R$ and a point $\xi^\kappa + d\xi^\kappa$ in R to a point $'\xi^\kappa + d'\xi^\kappa$ in $'R$.

---

[1] In principle, we follow, throughout the book, the standard notations which appear in the recent book by SCHOUTEN [8]. The number in parentheses refers to the Bibliography at the end of the book.

[2] A function is said to be of *class* C^r in some region if it is continuous and has continuous derivatives with respect to the coordinates up to the order r at each point of the region, and it is said to be of *class* C^ω if it is analytic. A space is said to be of class C^r (C^ω) if it can be covered by a set of coordinate neighbourhoods in such a way that the transformation of coordinates in an overlapping domain is represented by functions of class C^r (C^ω) in that domain.

[3] We adopt the *summation convention*: If an index appears twice in a term once as a subscript and once as superscript, summation has to be effected on the range of the index.

[4] The $g_{\lambda\kappa}(\xi)$ means the value of $g_{\lambda\kappa}$ at the point ξ whose coordinates with respect to (κ) are ξ^κ. The $f^\kappa(\xi^\nu)$ in (1.2) denotes n functions of coordinates ξ^ν.

[5] A point transformation is said to be of class C^r (C^ω) if the functions defining it are of class C^r (C^ω).

If the distance $d's$ between two displaced points $'\xi^{\varkappa}$ and $'\xi^{\varkappa} + d'\xi^{\varkappa}$ is always equal to the distance between the two original points $\xi^{\varkappa}$ and $\xi^{\varkappa} + d\xi^{\varkappa}$, the point transformation (1.2) is called a *motion* [1] or an *isometry* in the V_n.

Now in order to formulate the condition for (1.2) to be a motion in a V_n, we proceed as follows:

The point transformation T carries a point $\xi^{\varkappa}$ in R to a point $'\xi^{\varkappa}$ in $'R$ and consequently the point transformation T^{-1} inverse to T carries the point $'\xi^{\varkappa}$ in $'R$ to the point $\xi^{\varkappa}$ in R. With this inverse point transformation $T^{-1} : {'\xi} \to \xi$, we can associate a coordinate transformation $(\varkappa) \to (\varkappa')$ such that the transform in R of a point in $'R$ by T^{-1} has the same coordinates with respect to $(\varkappa')$ as the original point in $'R$ had with respect to $(\varkappa)$. This coordinate transformation is given by the equation

$$(1.3) \qquad \xi^{\varkappa'} = {'\xi^{\varkappa}} \; ^{2}$$

that is

$$(1.4) \qquad \xi^{\varkappa'} = f^{\varkappa}(\xi^{\nu}).$$

This process $(\varkappa) \to (\varkappa')$ is called the *dragging along* of the coordinate system $(\varkappa)$ by the point transformation T^{-1}: $'\xi \to \xi$ and $(\varkappa')$ is called the *coordinate system dragged along by* T^{-1}.

By this dragging along of $(\varkappa)$ the $d'\xi^{\varkappa}$ at $'\xi^{\varkappa}$ becomes $d\xi^{\varkappa'}$ at $\xi^{\varkappa'}$ and we have

$$(1.5) \qquad d\xi^{\varkappa'} = d'\xi^{\varkappa}.$$

Now the distance $d's$ between $'\xi^{\varkappa}$ and $'\xi^{\varkappa} + d'\xi^{\varkappa}$ is given by

$$(1.6) \qquad d's^2 = g_{\lambda\varkappa}({'\xi}) d'\xi^{\lambda} \, d'\xi^{\varkappa}$$

and the distance ds between $\xi^{\varkappa}$ and $\xi^{\varkappa} + d\xi^{\varkappa}$ is given by (1.1). But in the coordinate system $(\varkappa')$, (1.1) can be written as

$$(1.7) \qquad ds^2 = g_{\lambda'\varkappa'}(\xi) d\xi^{\lambda'} \, d\xi^{\varkappa'}$$

where

$$(1.8) \qquad g_{\lambda'\varkappa'}(\xi) = A^{\lambda\varkappa}_{\lambda'\varkappa'} g_{\lambda\varkappa}(\xi) \; ^{3}.$$

---

[1] Following this definition, the reflexion is a motion.

[2] Cf. SCHOUTEN [8], p. 102. This is written more elaborately $\xi^{\varkappa'} = \delta^{\varkappa'}_{\varkappa} {'\xi^{\varkappa}}$ where $\delta^{\varkappa'}_{\varkappa}$ is the general Kronecker delta. In all cases where no ambiguity can arise, we drop the symbol $\delta^{\varkappa'}_{\varkappa}$ for the sake of shortness.

[3] $A^{\lambda\varkappa}_{\lambda'\varkappa'} \stackrel{\text{def}}{=} A^{\lambda}_{\lambda'} A^{\varkappa}_{\varkappa'}$ and $A^{\varkappa}_{\varkappa'} \stackrel{\text{def}}{=} \partial_{\varkappa'} \xi^{\varkappa}$, $A^{\varkappa'}_{\varkappa} \stackrel{\text{def}}{=} \partial_{\varkappa} \xi^{\varkappa'}$.

Thus comparing (1.6) with (1.7) and taking account of (1.5), we have

$$g_{\lambda\varkappa}('\xi) = g_{\lambda'\varkappa'}(\xi) \tag{1.9}$$

for a motion in the V_n.

Now the field $g_{\lambda\varkappa}(\xi)$ is given at each point ξ of the space and consequently we have the field $g_{\lambda\varkappa}('\xi)$ at $'\xi$ in $'R$. Starting from this field $g_{\lambda\varkappa}('\xi)$ at $'\xi$, we form a new field $'g_{\lambda\varkappa}(\xi)$ at ξ in R in the following way:

We define a new field $'g_{\lambda\varkappa}(\xi)$ at ξ in R as a field whose components $'g_{\lambda'\varkappa'}(\xi)$ with respect to $(\varkappa')$ at each point ξ in R are equal to the $g_{\lambda\varkappa}('\xi)$ at the corresponding point $'\xi$ in $'R$, that is,

$$'g_{\lambda'\varkappa'}(\xi) \stackrel{\text{def}}{=} g_{\lambda\varkappa}('\xi) \tag{1.10}$$

Since

$$'g_{\lambda\varkappa}(\xi) = A_{\lambda\ \varkappa}^{\lambda'\varkappa'}\, 'g_{\lambda'\varkappa'}(\xi),$$

we have from (1.4) and (1.10),

$$'g_{\lambda\varkappa}(\xi) = (\partial_\lambda f^\sigma)(\partial_\varkappa f^\rho) g_{\sigma\rho}('\xi). \tag{1.11}$$

This process $g_{\lambda\varkappa} \rightarrow {}'g_{\lambda\varkappa}$ is called the *dragging along* of the field $g_{\lambda\varkappa}$ by the point transformation T^{-1} and the field $'g_{\lambda\varkappa}$ is called the field *dragged along*. We say also that the point transformation T^{-1} has *deformed* the tensor $g_{\lambda\varkappa}$ into $'g_{\lambda\varkappa}$ and we call $'g_{\lambda\varkappa}$ the *deformed tensor* of $g_{\lambda\varkappa}$ by T^{-1}.

Now comparing (1.9) with (1.10) we have

$$'g_{\lambda'\varkappa'}(\xi) = g_{\lambda'\varkappa'}(\xi) \tag{1.12}$$

with respect to $(\varkappa')$ and

$$'g_{\lambda\varkappa}(\xi) = g_{\lambda\varkappa}(\xi) \tag{1.13}$$

with respect to $(\varkappa)$ for a motion in V_n. Hence we have

THEOREM 1.1. *In order that (1.2) be a motion in a V_n, it is necessary and sufficient that the transformation $'\xi \rightarrow \xi$ do not deform the fundamental tensor of the V_n.*

We call $'g_{\lambda\varkappa} - g_{\lambda\varkappa}$ the *Lie difference* of $g_{\lambda\varkappa}$ with respect to (1.2). The Lie difference of $g_{\lambda\varkappa}$ is a tensor of the same type as $g_{\lambda\varkappa}$, because it is the difference of two tensors of this type. In order that (1.2) be a motion in a V_n, it is necessary and sufficient that the Lie difference of the fundamental tensor of V_n with respect to (1.2) vanish.

We now consider the case in which the point transformation (1.2) is an infinitesimal one

$$'\xi^\varkappa = \xi^\varkappa + v^\varkappa dt, \tag{1.14}$$

where $v^\varkappa$ is a contravariant vector field and dt is an infinitesimal. For the coordinate transformation (1.4) we have

(1.15) $$\xi^{\varkappa'} = f^\varkappa(\xi^\nu) = \xi^\varkappa + v^\varkappa dt,$$

from which

(1.16) $$\partial_\lambda f^\varkappa = \delta_\lambda^\varkappa + \partial_\lambda v^\varkappa dt$$

up to infinitesimals of the first order with respect to dt. In the following we shall always neglect quantities of an order higher than the first with respect to dt. Of course the equalities (1.14) and (1.16) should be written with the use of the sign * [1] because they are only valid for special coordinate systems. But we may accept as a general rule that * will be dropped in cases where no ambiguity can arise.

Substituting (1.16) in (1.11), we find

$$'g_{\lambda\varkappa} = (\delta_\lambda^\sigma + \partial_\lambda v^\sigma dt)(\delta_\varkappa^\rho + \partial_\varkappa v^\rho dt)(g_{\sigma\rho} + v^\mu \partial_\mu g_{\sigma\rho} dt),$$

from which

(1.17) $$'g_{\lambda\varkappa} = g_{\lambda\varkappa} + (v^\mu \partial_\mu g_{\lambda\varkappa} + g_{\rho\varkappa} \partial_\lambda v^\rho + g_{\lambda\rho} \partial_\varkappa v^\rho) dt.$$

Thus we have

THEOREM 1.2. *In order that* (1.14) *be a motion in a* V_n, *it is necessary and sufficient that*

(1.18) $$v^\mu \partial_\mu g_{\lambda\varkappa} + g_{\rho\varkappa} \partial_\lambda v^\rho + g_{\lambda\rho} \partial_\varkappa v^\rho = 0.$$

We call

(1.19) $$\underset{v}{\pounds} g_{\lambda\varkappa} dt \overset{\text{def}}{=} {}'g_{\lambda\varkappa} - g_{\lambda\varkappa} \; [2]$$
$$= (v^\mu \partial_\mu g_{\lambda\varkappa} + g_{\rho\varkappa} \partial_\lambda v^\rho + g_{\lambda\rho} \partial_\varkappa v^\rho) dt$$

---

[1] The sign * is used to emphasize the fact that an equation is only valid or that its validity is only asserted for the coordinate system or coordinate systems occuring explicitly in the formula itself. Cf. SCHOUTEN [8], p. 2.

[2] In the coordinate system $(\varkappa')$ which only differs infinitesimally from $(\varkappa)$, this equation can be written as

$$\underset{v}{\pounds} g_{\lambda'\varkappa'} dt = {}'g_{\lambda'\varkappa'}(\xi) - g_{\lambda'\varkappa'}(\xi) = g_{\lambda\varkappa}('\xi) - g_{\lambda'\varkappa'}(\xi).$$

But as is stated below, $\underset{v}{\pounds} g_{\lambda\varkappa}$ is a tensor and consequently

$$\underset{v}{\pounds} g_{\lambda'\varkappa'} dt = A_{\lambda'\varkappa'}^{\lambda\varkappa} \underset{v}{\pounds} g_{\lambda\varkappa} dt = \underset{v}{\pounds} g_{\lambda\varkappa} dt + (\text{term of higher order}),$$

from which

$$\underset{v}{\pounds} g_{\lambda\varkappa} dt = g_{\lambda\varkappa}('\xi) - g_{\lambda'\varkappa'}(\xi).$$

This is the usual definition of the Lie derivative. See YANO [13].

the *Lie differential* of $g_{\lambda\varkappa}$ with respect to (1.14) or with respect to the vector field $v^\varkappa$ and $\underset{v}{\pounds}\, g_{\lambda\varkappa}$ the *Lie derivative* [1] of $g_{\lambda\varkappa}$.

The Lie differential of $g_{\lambda\varkappa}$ is a tensor of the same type as $g_{\lambda\varkappa}$. Thus the Lie derivative of $g_{\lambda\varkappa}$ is also a tensor of the same type.

In fact, using the relations

$$\nabla_\mu g_{\lambda\varkappa} \overset{\text{def}}{=} \partial_\mu g_{\lambda\varkappa} - g_{\rho\varkappa}\{^{\rho}_{\mu\lambda}\} - g_{\lambda\rho}\{^{\rho}_{\mu\varkappa}\} = 0, \text{ [2] [3]}$$
$$\nabla_\mu v^\varkappa \overset{\text{def}}{=} \partial_\mu v^\varkappa + \{^{\varkappa}_{\mu\lambda}\} v^\lambda,$$

we can write the Lie derivative of $g_{\lambda\varkappa}$ in the form

$$(1.20) \qquad \boxed{\underset{v}{\pounds}\, g_{\lambda\varkappa} = 2\nabla_{(\lambda} v_{\varkappa)} \text{ [4]}} \; ; \quad v_\varkappa \overset{\text{def}}{=} v^\lambda g_{\lambda\varkappa}, \text{ [5]}$$

which shows explicitly the tensor character of $\underset{v}{\pounds}\, g_{\lambda\varkappa}$.

Thus we have

THEOREM 1.3. *In order that* (1.14) *be a motion in a* V_n *it is necessary and sufficient that the Lie derivative of* $g_{\lambda\varkappa}$ *with respect to* (1.14) *vanish:*

$$(1.22) \qquad \underset{v}{\pounds}\, g_{\lambda\varkappa} = 2\nabla_{(\lambda} v_{\varkappa)} = 0.$$

The equation (1.22) is called after Killing [6] and a vector field satisfying a Killing equation is called a *Killing vector.*

Myers and Steenrod [7] proved

THEOREM 1.4. *Any closed group of motions in a* V_n *of class* C^r $(r \geq 2)$ *is a Lie group of motions.*

---

[1] The name "Lie derivative" was introduced by VAN DANTZIG [2, 3].

[2] We use the notations $\delta\Phi$ and $\nabla_\mu\Phi$ to denote the covariant differential and the covariant derivative of Φ respectively. Cf. SCHOUTEN [8], p. 124.

[3] The $\{^{\varkappa}_{\mu\lambda}\}$ denotes the Christoffel symbol: $\{^{\varkappa}_{\mu\lambda}\} \overset{\text{def}}{=} \frac{1}{2} g^{\varkappa\rho}(\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda})$. Cf. SCHOUTEN [8], p. 132.

[4] The round brackets denote the symmetric part, e.g. $2\nabla_{(\lambda} v_{\varkappa)} = \nabla_\lambda v_\varkappa + \nabla_\varkappa v_\lambda$, while the square brackets denote the alternating part, e.g. $\Gamma^\varkappa_{[\mu\lambda]} = \frac{1}{2}(\Gamma^\varkappa_{\mu\lambda} - \Gamma^\varkappa_{\lambda\mu})$. Cf. SCHOUTEN [8], p. 14.

[5] In the following we distinguish the contravariant, covariant and mixed components of a tensor by the position of the indices, the same kernel being used in all cases. Cf. SCHOUTEN [8], p. 44.

[6] KILLING [1].

[7] MYERS and STEENROD [1].

§ 2. Affine motions in a space with a linear connexion.

We consider in this section n-dimensional space L_n [1] provided with a linear connexion $\Gamma^{\varkappa}_{\mu\lambda}(\xi)$. In an L_n the parallelism between a vector $u^{\varkappa}$ at a point $\xi^{\varkappa}$ and a vector $u^{\varkappa} + du^{\varkappa}$ at a point $\xi^{\varkappa} + d\xi^{\varkappa}$ is defined by

(2.1) $$\delta u^{\varkappa} \overset{\text{def}}{=} du^{\varkappa} + \Gamma^{\varkappa}_{\mu\lambda} u^{\lambda} d\xi^{\mu} = 0.$$

When we effect a point transformation (1.2), the differentials $d\xi^{\varkappa}$ at $\xi^{\varkappa}$ are transformed into the differentials

(2.2) $$d'\xi^{\varkappa} = \frac{\partial f^{\varkappa}}{\partial \xi^{\nu}} d\xi^{\nu}$$

at $'\xi^{\varkappa}$. Now if we make the condition that the vector $u^{\varkappa}$ at $\xi^{\varkappa}$ is transformed from $\xi^{\varkappa}$ to $'\xi^{\varkappa}$ in the same way as the linear elements $d\xi^{\varkappa}$ at $\xi^{\varkappa}$, then the corresponding vector at $'\xi$ is

(2.3) $$\overset{m}{u}{}^{\varkappa}('\xi) = \frac{\partial f^{\varkappa}}{\partial \xi^{\nu}} u^{\nu}(\xi).$$

When a point transformation (1.2) transforms any pair of parallel vectors into a pair of parallel vectors, (1.2) is called an *affine motion* [2] in an L_n.

For an affine motion, we must have

(2.4) $$\delta \overset{m}{u}{}^{\varkappa}('\xi) \overset{\text{def}}{=} d\overset{m}{u}{}^{\varkappa}('\xi) + \Gamma^{\varkappa}_{\mu\lambda}('\xi)\overset{m}{u}{}^{\lambda}('\xi) d'\xi^{\mu} = 0.$$

Now we introduce the coordinate transformation $\xi^{\varkappa'} = {}'\xi^{\varkappa}$. Then with respect to $(\varkappa')$ dragged along by $T^{-1} : {}'\xi \to \xi$, the equation (2.1) can be written as

(2.5) $$\delta u^{\varkappa'} \overset{\text{def}}{=} du^{\varkappa'}(\xi) + \Gamma^{\varkappa'}_{\mu'\lambda'}(\xi) u^{\lambda'}(\xi) d\xi^{\mu'} = 0,$$

where

(2.6) $$u^{\varkappa'}(\xi) = A^{\varkappa'}_{\varkappa} u^{\varkappa}(\xi)$$

and

(2.7) $$\Gamma^{\varkappa'}_{\mu'\lambda'}(\xi) = (A^{\mu\lambda}_{\mu'\lambda'} \Gamma^{\varkappa}_{\mu\lambda}(\xi) + \partial_{\mu'} A^{\varkappa}_{\lambda'}) A^{\varkappa'}_{\varkappa},$$

and (2.3) can now be written as

(2.8) $$\overset{m}{u}{}^{\varkappa}('\xi) = u^{\varkappa'}(\xi).$$

---

[1] An n-dimensional space with a linear connexion is called an L_n. Cf. SCHOUTEN [8], p. 125.

[2] An affine motion was first defined by SLEBODZINSKI [2].