

**Signal Analysis
and Estimation**
An Introduction

Ronald L. Fante

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Wilmington, Massachusetts



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*To the honor and glory of God
and the memory of my brother
Mel Fante*

Signal Analysis and Estimation

PREFACE

This text is intended as an introduction to the analysis (using Fourier techniques) of continuous and discrete deterministic signals, along with both the estimation and spectral analysis of random signals. Because the treatment is applicable to optical, acoustical, electrical, etc., signals, it should be useful to students of both physics and engineering (especially electrical).

In order to be appropriate for a course given at any time in the curriculum, the presentation has been designed so that no detailed mathematical knowledge beyond the completion of sophomore calculus is required. No familiarity with Fourier transforms is assumed, nor is any extensive knowledge of probability and random processes. Furthermore, vector calculus and complex variables have been avoided wherever possible. Also, matrix notation is used sparingly, although a limited familiarity with matrix operations is assumed.

The text is designed for a 12-week course and divided into two portions: deterministic signals and random signals. Chapters 1–5 are devoted to the analysis of continuous and discrete deterministic signals, and it is expected that this material could be covered in approximately 5 weeks. The properties, spectral analysis and estimation of random signals are covered in Chapters 6–9, and should occupy the remainder of the course.

In order to assist the reader in understanding the (sometimes complex) concepts presented, examples have been liberally included in every chapter. Usually, those examples chosen are related to engineering applications, such as radar, antennas, optical processing, tomography, analog and digital filters, and target tracking, so that the student can understand how the techniques studied are applied in practice. Also, the problems are designed not only to test the readers understanding, but also to illustrate applications and extensions that could not be included in the main text.

The question of notation is always a vexing one. It is, of course, impossible to find a system of notation that is consistent with that used by all other authors. I have tried to use conventional notation wherever possible, but there were several instances where I chose to depart from usual electrical engineering notation. In particular, I use a prime on a symbol to indicate that it represents an estimate, whereas many authors use a caret over the symbol for this purpose. Instead, I have used a caret over a symbol to indicate that it is the Fourier transform of a discrete signal (to differen-

tiate it from the Fourier transform of a continuous signal). The other departure from the usual notation of electrical engineering, but not physics, is my use of an overbar (i.e., \bar{x} = expectation of x) to denote an expectation (or ensemble average) of a random variable, whereas many authors use $E(x)$ for this purpose. I was forced to abandon $E(x)$ in favor of \bar{x} because many of the results in Chapters 7–9 became too cumbersome when the $E(\dots)$ notation was used.

Finally, I wish to express my thanks to Avco/Textron for providing an environment conducive to the preparation of this manuscript, and especially to Clara for her continuous support and encouragement.

*Wilmington, Massachusetts
September 1987*

RONALD L. FANTE

INTRODUCTION

Fourier methods will be used here to study the properties of both deterministic and random signals. Although other techniques are sometimes used, Fourier analysis is by far the most common and useful, having important applications in virtually every field of engineering and physics. Some examples are

1. *Signal Analysis*. The Fourier transform allows one to deduce the frequency content (i.e., how fast the signal varies) of an arbitrary signal.
2. *Quantum Mechanics*. The position and momentum of a particle are a Fourier transform pair; the Heisenberg uncertainty principle expresses a property common to all Fourier transform pairs.
3. *Linear Systems*. The analysis and design of both continuous and discrete linear systems is considerably simplified through use of Fourier transforms. In particular, it is found that the Fourier transform of the output of an arbitrary linear system is simply the product of the Fourier transform of the input to the system and a function (called the system function) that describes the linear system.
4. *Antennas*. The signal radiated by either an electromagnetic or acoustic antenna is equal to the Fourier transform of the source distribution.
5. *Optical Processors*. The electric field distribution in an appropriate plane behind a thin lens is the Fourier transform of the field in an appropriate plane in front of the lens. This property forms the basis of coherent optical processing and is a cornerstone of the computer of the future.
6. *Spectroscopy*. There is a technique known as “Fourier Transform Spectroscopy” where Fourier transforms are used to determine the spectral content (brightness versus wavelength) of an arbitrary light source.
7. *Tomography*. A special property of two-dimensional Fourier transforms forms the basis of tomographic image reconstruction.
8. *Partial Differential Equations*. Fourier transforms are routinely used in the solution of partial differential equations and other areas of mathematical physics.

Thus, Fourier transforms are quite important and the discovery of fast algorithms for computing these transforms has made them an even more valuable and useful tool.

Part I of this text will be devoted to the study of deterministic signals. In Chapter 1 it is shown how Fourier series can be used to represent and analyze periodic functions. Then after a digression (Chapter 2) on the properties of the singularity functions (Dirac delta function, etc.), Chapter 3 covers the application of the Fourier transform to the analysis of aperiodic continuous (analog) signals. Chapter 4 shows how one obtains discrete signals from continuous ones, and the role of the Fourier transform in the analysis of digital signals. Chapter 5 discusses the discrete Fourier transform and special algorithms that are appropriate for their rapid computer-computation.

These approaches are applicable to deterministic signals. For random signals the approach must be generalized, and Part II is devoted to this end. Chapter 6 is a relatively extensive discussion of the properties and representation of random signals, and in Chapter 7 we show how Fourier techniques are used to determine their spectral (frequency) content. Chapter 8 is devoted to the principles of signal estimation, and in Chapter 9 these techniques are used to estimate the power spectral density of random signals.

CONTENTS

1. Fourier series	3
1.1 The Fourier sine/cosine series / 3	
1.2 The complex Fourier series / 12	
1.3 Parseval's theorem / 14	
1.4 Proof that Fourier series give best mean-square fit to periodic function / 15	
1.5 Product of periodic functions / 18	
1.6 Periodic amplitude modulation / 19	
1.7 Periodic phase modulation of harmonic carrier / 20	
1.8 The Fourier series in two dimensions / 28	
References / 32	
Problems / 32	
2. The singularity functions	40
2.1 Definition of the Dirac delta function / 40	
2.2 An important result / 45	
2.3 Properties of the delta function / 46	
2.4 Two-dimensional delta function / 49	
References / 54	
Problems / 54	
3. The Fourier transform	56
3.1 Definition of the Fourier transform / 56	
3.2 Fourier transform properties / 63	
3.3 Fourier transform of the singularity functions / 70	
3.4 Products / 75	
3.5 Truncating or windowing a signal / 82	
3.6 Linear systems and convolution / 87	

3.7	Narrow-band signals / 92	
3.8	Two-dimensional Fourier transform / 95	
3.9	Polar form of the two-dimensional transform / 97	
3.10	Rotations and projections / 100	
3.11	Tomography / 102	
3.12	Optical processing / 105	
3.13	Solving differential equations with Fourier transforms / 109	
	References / 111	
	Problems / 112	
4.	Discrete signals and their Fourier transform	126
	Introduction / 126	
4.1	Bandlimited signals / 126	
4.2	Uniform sampling of bandlimited functions / 128	
4.3	Reconstruction from sampled data / 136	
4.4	Sampling errors / 140	
4.5	Frequency analysis of discrete signals / 144	
4.6	Discrete linear systems / 153	
4.7	The Z-transform / 158	
4.8	Two-dimensional discrete signals / 160	
4.9	Determining missing samples / 168	
	References / 170	
	Problems / 170	
5.	The discrete Fourier transform and fast computation algorithms	175
	Introduction / 175	
5.1	Windowing of discrete data / 175	
5.2	Sampling $\hat{W}(f)$ / 182	
5.3	The discrete Fourier transform / 185	
5.4	Properties of the discrete Fourier transform / 191	
5.5	Recovering the continuous Fourier transform / 203	
5.6	Relation of discrete to continuous convolution / 204	
5.7	The fast Fourier transform / 208	
5.8	The discrete Fourier transform as a non-recursive filter / 215	
5.9	The two-dimensional discrete Fourier transform / 217	
	References / 218	
	Appendix 1: Zero padding the discrete Fourier transform / 218	
	Appendix 2: Relationship of discrete and continuous transforms / 220	
	Problems / 221	

6. Introduction to random signals	229
Introduction / 229	
6.1 Elementary probability theory / 230	
6.2 Entropy / 242	
6.3 Higher-order probabilities / 244	
6.4 Higher-order ensemble averages / 250	
6.5 Correlations / 256	
6.6 Time averages and the ergodic theorem / 266	
6.7 Effect of linear systems on correlation and entropy / 272	
6.8 The gaussian (normal) process / 279	
6.9 Functions of random variables / 290	
References / 297	
Problems / 298	
7. Spectral analysis of random signals	307
Introduction / 229	
7.1 Power spectral density / 307	
7.2 Wiener-Khintchine theorem / 310	
7.3 Cross-spectral density / 317	
7.4 Reconstruction theorem for sampled random signals / 321	
7.5 Linear systems / 322	
7.6 The matched filter / 327	
7.7 Narrow-band random signals / 331	
7.8 Multidimensional spectral density / 341	
References / 343	
Problems / 344	
8. Principles of estimation theory	349
Introduction / 349	
8.1 Fundamentals / 350	
8.2 Bayes (nonlinear) estimators / 357	
8.3 Linear mean-squared estimation / 370	
8.4 Recursive linear estimation / 378	
8.5 Linear one-step prediction / 387	
8.6 Parametric estimation / 391	
References / 397	
Appendix: Derivation of the Cramer-Rao bound / 398	
Problems / 399	

9. Spectral estimation	406
Introduction /	406
9.1 The periodogram /	406
9.2 Periodogram averaging (Bartlett's method) /	411
9.3 Periodogram smoothing /	414
9.4 Spectral estimation with short data records: introduction /	419
9.5 Rational-transfer-function model: deterministic data /	420
9.6 Maximum entropy method /	429
9.7 Rational-transfer-function model: random data /	433
References /	437
Appendix: Variance of the smoothed periodogram /	438
Problems /	440
Index	445

PART I

DETERMINISTIC SIGNALS

1

Fourier Series

1.1 THE FOURIER SINE/COSINE SERIES

We shall begin our discussion of signal analysis by considering some simple periodic signals such as shown in Figure 1.1 (these are typical of the pulse trains produced by an electronic signal generator for use as a clock in a digital system). A signal $s(t)$ is called “periodic” in time t if it repeats itself every T seconds, so that $s(t + T) = s(t)$. The quantity T is called the period of the signal, and $F = 1/T$ is known as its frequency. A quantity $\omega = 2\pi \cdot \text{frequency}$ is also often used and is known as the “radian frequency.” The units for the period are “seconds,” those for the frequency are usually “hertz” (abbreviated Hz) or sec^{-1} , and those for the radian frequency are “radians per second.”

Upon observing the signal $s_1(t)$ in Figure 1.1a, one might guess that, without too great an error, it could be approximated by $\cos(2\pi t/T)$, which is shown as a dashed line. It might also be expected that the fit would be ever better if more cosines are added and $s_1(t)$ approximated by

$$s_1(t) = A_1 \cos\left(\frac{2\pi t}{T}\right) + A_2 \cos 2\left(\frac{2\pi t}{T}\right) + \cdots + A_n \cos n\left(\frac{2\pi t}{T}\right), \quad (1)$$

where the coefficients are chosen (in a way to be discussed later) to provide the best possible fit to the actual signal. Finally, we could speculate that if we kept enough terms in Eq. (1) the fit would become perfect. This can be shown to be precisely what happens (except for very near any discontinuities in $s_1(t)$, where there is a mismatch known as the Gibbs phenomenon; this will be discussed later).

The series in Eq. (1) contains only cosines because the function $s_1(t)$ is symmetric about the time origin (a symmetric function is one for which $s(-t) = s(t)$). Now suppose we displace $s_1(t)$ to the right by a time Δ to produce the function $s_2(t)$ shown in Figure 1.1b. Then in all the terms on the right-hand side of Eq. (1) t is replaced by $t - \Delta$ and the n th term becomes

$$\cos \frac{2\pi n}{T} (t - \Delta) = \cos \frac{2\pi n t}{T} \cos \frac{2\pi n \Delta}{T} + \sin \frac{2\pi n t}{T} \sin \frac{2\pi n \Delta}{T}, \quad (2)$$

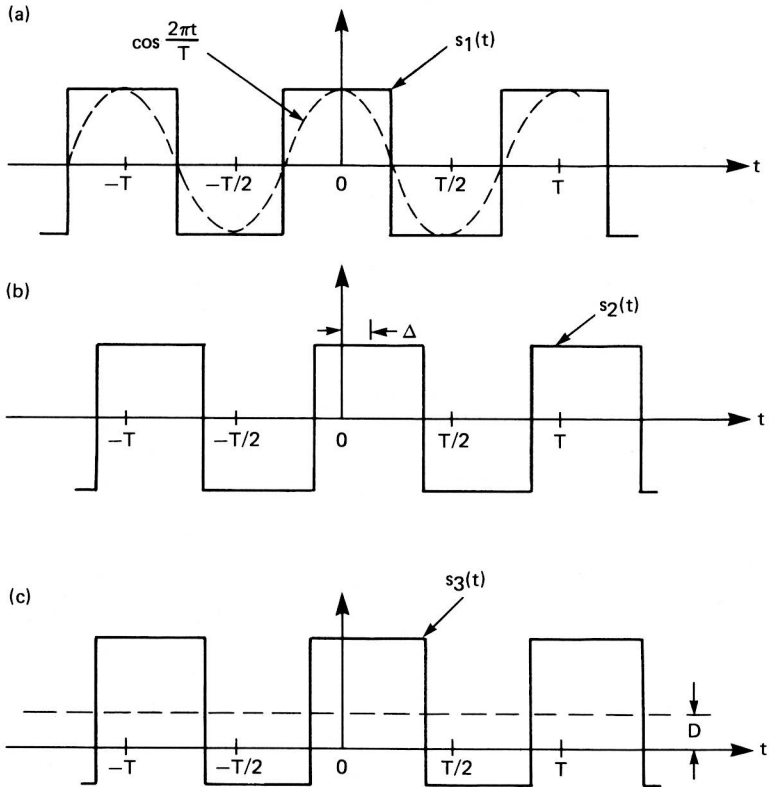


Figure 1.1 Periodic pulse trains.

so that the result in Eq. (1) now is

$$s_2(t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{2\pi n \Delta}{T} \right] \cos \frac{2\pi n t}{T} + \sum_{n=1}^{\infty} \left[A_n \sin \frac{2\pi n \Delta}{T} \right] \sin \frac{2\pi n t}{T} \quad (3)$$

$$= \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}, \quad (4)$$

where $a_n = A_n \cos(2\pi n \Delta / T)$ and $b_n = A_n \sin(2\pi n \Delta / T)$.

Finally, suppose that the function $s_2(t)$ is displaced upward by an amount D to produce $s_3(t)$ as shown in Figure 1.1c. Then a constant term $a_0/2 = D$ would need to be added to the series in Eq. (4), and this gives

$$s_3(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n t}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2\pi n t}{T}. \quad (5)$$