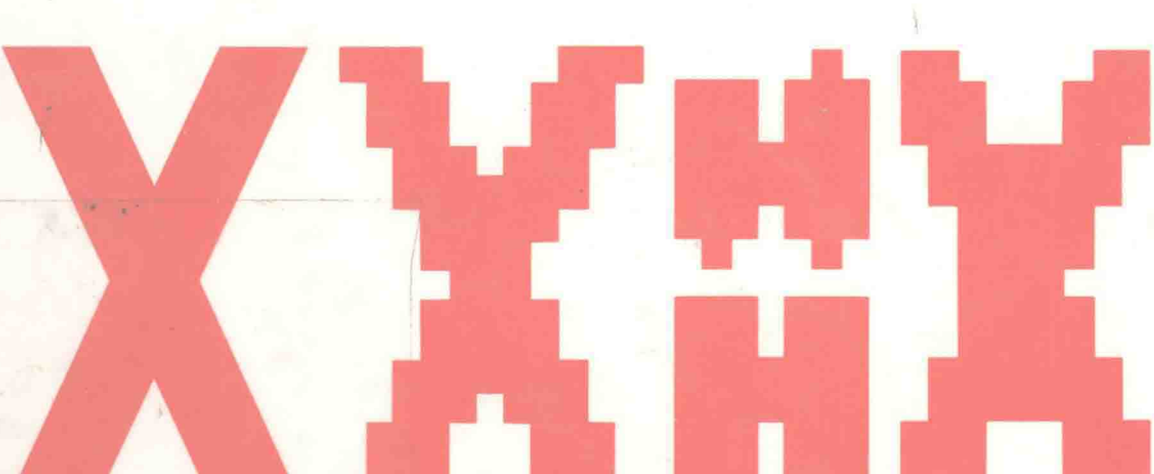
THIRD EDITION

APPLIED LINEAR ALGEBRA



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Third edition

Applied linear algebra

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To Denise, Anna, and John Ben

To Adam and Joshua, and Ann and Susan

Preface

Overall approach

Linear algebra is an essential part of the mathematical toolkit required in the modern study of many areas in the behavioral, natural, physical, and social sciences, in engineering, in business, in computer science, and of course in pure and applied mathematics. Our aim in this book is to develop the fundamental concepts of linear algebra, emphasizing those concepts that are most important in applications, and to illustrate the applicability of these concepts through a variety of applications. Thus, while we present applications for illustrative and motivative purposes, our main goal is to present mathematics that can be applied.

We have taken great care in presenting the theory of linear algebra rather fully, although from as concrete a viewpoint as possible. We therefore begin with the concrete manipulative matrix and vector algebra (Chapter 1) and with Gauss elimination (Chapter 3) before the *theory* of linear equations (Chapter 4) or the abstract notions of vector spaces (Chapter 5) and linear transformations (Chapter 6). Similarly, eigensystems and various related decompositions and canonical forms are presented (Chapter 7 and, in more detail, Chapters 8 and 9) as tools for simplifying the study of linear transformations modeling the behavior of complicated systems.

In addition to presenting a number of applications briefly throughout the text, we have collected in Chapter 2 a special set of applications to motivate the later material; only familiarity with the basic matrix algebra of Chapter 1 is assumed, and we not only show how matrices arise in practice but also raise questions about matrices and their properties that stimulate interest in the ideas to come. Later chapters often refer to these examples for motivation, although the remainder of the book is in fact independent of Chapter 2 for those who prefer to omit it.

The entire book builds upon the notion of elementary row operations and Gauss elimination. The manipulative approach developed in Chapter 3 is fundamental to almost all the theoretical as well as applied topics. It is *absolutely essential* for the student to learn these techniques.

New features in the third edition

We created this new edition with the primary goal of making it easier to teach and to learn from the book. Except for some additions, the content is approximately the same as in the second edition, but the presentation has been reworked for a more consistent style and approach. Triangular forms produced by Gauss elimination often replace the (reduced-) row-echelon forms emphasized for theoretical work in the prior editions.

The problem sets have been expanded. This edition contains over 1100 problems, compared with just over 600 in the second edition. Problem sets appear at the end of each section instead of being scattered throughout the text as Exercises (a mix of examples and problems). An appendix contains answers or suggestions for over one-third of the problems (those indicated by a ▷ symbol preceding the problem numbers).

We now number all displayed material consecutively, so that displayed statement (4.15) falls immediately between Example 4.14 and **Key Theorem 4.16**, for example. Guidelines appear in most longer sections, indicating which of the problems can be solved using material presented by that point. A brief refresher on complex numbers appears early in Chapter 1; results are usually stated separately for the real and complex cases for those wishing to concentrate on one or the other.

In this edition many topics appear in the main text rather than merely in the problems. Greater emphasis is thus given to the Cayley-Hamilton theorem, Perron-Frobenius theorem, *LU*-decomposition, orthogonal projections and projection matrices, best approximation, fundamental frequencies of oscillating systems, and so on. New material has also been included, introducing such topics as adjoint transformations, Karmarkar-like methods for linear programming, theory of linear inequalities, convergence of sequences of vectors and matrices, and isomorphisms.

Examples and some problems (about 6%) have been added that make explicit use of computer software for matrix computations. The book can still be used completely independently of computers, but those wishing to take advantage of microcomputer and mainframe software will find it easy to do so. For more information on this, see the material following this Preface.

We realize that this text in its previous two editions has been used for a variety of courses. The new edition is designed to continue this option, which of course places a special responsibility on the instructor to select the appropriate material. Some guidelines on this now follow. Those interested in the possible support of this text through the use of computers should also examine the section after this Preface.

Use for elementary theoretical linear algebra courses

Those wishing to emphasize proofs in a classic theorem-proof approach can easily do so. Theory is fundamental to applied linear algebra and is therefore presented with great care (and affection). The standard one-semester sophomore course would cover Chapter 1 on matrix algebra, Sections 1 through 5 of Chapter 3 on techniques for equations and inverses, most of Chapter 4 on theory for equations and inverses (including determinants), Sections 1 through 5, 7, and 8 of Chapter 5 on vector spaces, the first two sections on linear transformations in Chapter 6, and possibly the first three sections on eigensystems in Chapter 7. We certainly hope that such courses will also include some application-oriented material: a section from Chapter 2 illustrating an application, the LU-decomposition of Section 3.7, and norms and least squares in Sections 5.6 and 5.9 are good possibilities.

Use for elementary applied linear algebra courses

Along with their intermediate/advanced versions, introductory applied linear algebra courses have been the main contexts in which the previous two editions of this book have been used for nearly twenty years. Many variations in these courses are possible, depending on such aspects as which application areas are to be emphasized (engineering? social science? et cetera) and how much proving is to be done (none? selected **Key Theorems**? et cetera). We sketch a basic syllabus that provides a good mix.

Chapter 1 on matrix algebra should be carefully covered, followed by two applications from Chapter 2 (we choose those that connect with topics emphasized later). We would then proceed carefully through Chapter 3 on methods for equations and inverses, followed by a light treatment (especially regarding determinants) of the theory in Chapter 4; most time would be spent explaining the theory, with an occasional instructive proof presented in detail. By emphasizing the facts rather than the proofs, we would cover vector spaces and linear transformations in Chapters 5 and 6 (most sections); plenty of time must be allowed for students to grasp the concepts of linear independence and dependence. Since Chapter 7 gives a broad overview of the theory and use of eigensystems, we would generally cover this rather than the more detailed and specialized Chapters 8 or 9; those chapters are good alternatives, however, for those wishing to stress particular types of applications. Depending on the availability of time, we might introduce some topics from later chapters: least squares (Section 8.5), differential equations (Section 9.5), quadratic forms (Section 10.3), or linear programs (Section 11.1).

This covers quite a bit of material. The key to doing so is to stress the *concepts* and *techniques*; with this approach material can be covered quite quickly.

Use for subsequent applied linear algebra courses

For many courses our text has been used by students who have already had an introduction to linear algebra—often in a theory-oriented course. In such courses, two or three topics from Chapter 2 can be used as early motivation, followed by a quick review of Chapters 1, 3, and 4 (how quick depends on how much one assumes the students recall); the later sections of Chapter 3 on the practical aspects of solving equations—particularly the LU-decomposition—are rarely familiar to students from traditional courses and should be covered thoroughly. After briefly reviewing the basic information about vector spaces in the first half of Chapter 5, we would concentrate on the material there regarding norms, inner products, projections, and least squares. Likewise for Chapter 6—material on norms and perturbations is likely to be new.

The heart of such a course should be constructed from Chapters 7 through 11, again depending strongly on the emphasis desired and whether the students learned anything about eigensystems in their previous linear algebra courses.

Ben Noble and James W. Daniel

On the use of computers

We do *not* assume that either students or instructors will use computers to support their learning and teaching of linear algebra. As has always been true of its various versions, this book can be used independently of computing support and independently of the reader's interest in computing. We note, however, that each year more of the students in our classes are acquainted with computing and often are accustomed to using computers—mainframes or microcomputers—to assist with their course work in various ways including word processing, calculating, data analysis, and the like. This third edition of *Applied Linear Algebra* has been written to make it easier *for those who wish to do so* to support their learning and teaching with active involvement with computing.

Examples and Problems

To reflect the fact that scientific computation is now done on computers, many of our numbered Examples in the text are computer-based. *Such material is self-contained: we assume no prior computing knowledge on the part of student or instructor.* Our using computers simply allows us to consider and present far more realistic and interesting examples than if we had to restrict ourselves to contrived problems that can be solved entirely by hand. Such Examples are flagged in the margin by the symbol 𝔐.

Roughly 6% of our more than 1100 Problems are likewise flagged by the symbol 𝔐. Solving these problems requires access to computing, particularly to software designed to facilitate matrix computations. *Students and instructors should not write their own programs for solving linear algebra problems—the problems involve subtle aspects that are not obvious to the inexpert.* Superb software is available essentially for the asking: see Sections 3.9 and 7.6 on how to obtain high-quality programs for matrix computations. Since so many educational institutions now provide microcomputer centers for student use, we discuss in a bit more detail some excellent software available for this environment.

MATLAB

Our symbol 𝔐 stands for the initial letter in MATLAB, which itself stands for matrix laboratory. Over a number of years, MATLAB was developed by Cleve Moler, initially for use in teaching linear algebra and numerical analysis courses, using programs based on those from the LINPACK and EISPACK projects described in the text sections referenced previously. MATLAB was eventually released and widely distributed in a FORTRAN version for mainframe computers. Several commercial organizations developed enhanced versions for special applications such as control theory.

PC-MATLAB was the next evolutionary step—a highly optimized second-generation version of MATLAB developed especially for use by the IBM PC microcomputer. This in turn is being followed by other microcomputer versions of the system for use with other operating systems and other machines. For simplicity, we refer to the entire spectrum of software—from mainframe to the latest microcomputer versions—by the generic term "MATLAB."

We have personally found MATLAB to be very easy to learn (especially for the computer-wary), easy to use, and extremely powerful. Moreover, *MATLAB is available in special versions at greatly reduced cost for use in instruction*: contact The MathWorks, Inc., Suite 250, 20 North Main St., Sherborn, MA 01770, (617) 653-1415. One need not learn computer programming with this highly flexible and interactive system. Matrices are easily entered; typing

A = [1 2 3; 4 5 6; 7 8 9]

creates precisely the 3×3 matrix one would expect. Typing

inv(A)

produces its inverse,

eig(A)

its eigenvalues,

lu(A)

its LU-decomposition, and so on. It is therefore so easy to use that it is excellent for association with even a first linear algebra course. On the other hand, it is so powerful and flexible that it is an important tool for anyone whose work requires matrix calculations, whether they involve equations, eigensystems, least squares, data analysis, signal processing, or whatever.

We like MATLAB. We recommend it to you. And we have no financial interest in its success.

Contents

1

Matrix algebra

This first chapter is fundamental: its goal is to introduce matrices and those basic algebraic manipulations that the student must ***thoroughly*** *understand before proceeding. It is important to practice the addition and multiplication of matrices until these operations become automatic. Theorem 1.44 is a* ***Key Theorem****, providing the basis for later computational methods; the proof of Theorem 1.35 is a* ***Key Proof*** *because it illustrates a useful general argument.*

1.1 INTRODUCTION

Matrices provide convenient tools for systematizing laborious calculations by providing a compact notation for storing information and for describing complicated relationships.

(1.1) ***Definition.*** A $p \times q$ (read "p by q") *matrix* is a rectangular array $\mathbf{A}$ of pq numbers (or symbols representing numbers) enclosed in square brackets; the numbers in the array are called *entries* and are arranged in p horizontal *rows* and q vertical *columns.* The (i, j)-*entry* is denoted $\langle \mathbf{A} \rangle_{ij}$ and equals the entry in the ith row and jth column, numbering rows from top to bottom and columns from left to right; if $\mathbf{A}$ is $p \times 1$ (a *column matrix*) or $1 \times q$ (a *row matrix*), then $\langle \mathbf{A} \rangle_i$ is used instead of $\langle \mathbf{A} \rangle_{i1}$ or $\langle \mathbf{A} \rangle_{1i}$.

We denote matrices by boldface letters $\mathbf{A}$, $\mathbf{x}$, and so on. Using the helpful mnemonic device of denoting the (i, j)-entry of a matrix by a subscripted lowercase version of the boldface capital letter denoting the matrix, we write the general $p \times q$ matrix as

(1.2)
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ & \cdots & & \\ a_{p1} & a_{p2} & \cdots & a_{pq} \end{bmatrix}.$$

Matrix entries may be real or complex numbers; readers unsure about the basic properties of complex numbers can consult the brief refresher at the end of this page. Since matrices with various special structures often arise, we introduce some terminology to describe them.

(1.3) ***Definition***

(a) A *square matrix* **A** is $p \times q$ with $p = q$; the (i, i)-entries for $1 \leq i \leq p$ form the *main diagonal* of **A**.

(b) A *diagonal matrix* is a square matrix all of whose entries *not* on the main diagonal equal zero. By $\mathbf{diag}(d_1, \ldots, d_p)$ is meant the $p \times p$ diagonal matrix whose (i, i)-entry equals d_i for $1 \leq i \leq p$.

(c) A $p \times q$ *lower-triangular matrix* **L** satisfies $\langle \mathbf{L} \rangle_{ij} = 0$ if $i < j$, for $1 \leq i \leq p$ and $1 \leq j \leq q$.

(d) A $p \times q$ *upper-triangular matrix* **U** satisfies $\langle \mathbf{U} \rangle_{ij} = 0$ if $i > j$, for $1 \leq i \leq p$ and $1 \leq j \leq q$.

(e) A *unit-lower* (or *-upper*)*-triangular matrix* **T** is a lower (or upper)-triangular matrix satisfying $\langle \mathbf{T} \rangle_{ii} = 1$ for $1 \leq i \leq \min(p, q)$.

The use of matrix notation allows us to consider an array of many numbers as a single object denoted by a single symbol. Relationships among the large sets of numbers often arising in applications can then be expressed in a concise fashion. The more complicated the problem, the more useful matrices become. Perhaps more important, the use of matrices often provides insights that could not be obtained easily—if at all—by other means.

A Brief Refresher on Complex Numbers. Recall that a *complex number* has the form $a + b\imath$, where a and b are real numbers and $\imath$ denotes $\sqrt{-1}$. Thus $3 + 2\imath$, $-6 + 4.7\imath$, and $e - \pi\imath$ are complex numbers. So also are all real numbers r—since they equal $r + 0\imath$—and all *pure imaginary numbers* $t\imath$ for real t—since they equal $0 + t\imath$. Complex numbers are added and multiplied much like real numbers, but you must keep in mind that $\imath \cdot \imath = -1$. Thus

$$(3 - 2\imath)(-5 + 7\imath) = -15 + 21\imath + 10\imath - 14\imath\imath = -15 + 31\imath - 14(-1)$$
$$= -1 + 31\imath.$$

The *conjugate* $\bar{z}$ of the complex number $z = a + b\imath$ is the complex number $\bar{z} = a - b\imath$. The *magnitude* $|z|$ of the complex number $z = a + b\imath$ is defined as

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}.$$

Thus for $z = 3 - 2\imath$ we have

$$\bar{z} = 3 + 2\imath \quad \text{and} \quad |z| = \sqrt{13}.$$

PROBLEMS 1.1

▷ **1.** Let $z_1 = 2 - 3i$, $z_2 = -1 + 5i$, $z_3 = 2i$, and $z_4 = -7$. Evaluate the following, writing each answer in the standard form $a + bi$ with a and b real.

(a) $z_1 + 2z_2$ (b) $(1 + i)\bar{z}_1$ (c) $z_3 z_4$
(d) $\bar{z}_1 \bar{z}_3$ (e) $\overline{z_1 z_3}$ (f) $(z_2)^2$
(g) $(z_1)^3$ (h) $|z_1|$ (i) $|i\bar{z}_2|$
(j) $1/z_2$ (k) z_1/z_3

2. Prove that, for all complex numbers z and w, $\overline{zw} = \bar{z}\bar{w}$.

3. (a) How many entries are there in a $p \times q$ matrix?
(b) In its first row?
(c) In its last column?

▷ **4.** Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 6 & x \\ 2 & y & -3 \end{bmatrix}.$$

Give each of the following:

(a) the (1, 2)-entry of **A** (b) the (2, 2)-entry of **B**
(c) the (2, 2)-entry of **A** (d) $\langle \mathbf{A} \rangle_{21}$
(e) $\langle \mathbf{B} \rangle_{23}$ (f) $\langle \mathbf{A} \rangle_{32}$

5. Write the general 3×2 matrix **A** with entries a_{ij} in the form (1.2). Write out the general 2×3 matrix **B** with entries b_{ij} in the form (1.2).

6. Write the general 3×3 matrix of each type in Definition 1.3.

1.2 EQUALITY, ADDITION, AND MULTIPLICATION BY A SCALAR

In most applications, matrices must be combined in various ways much as numbers are combined through arithmetic. We in fact need concepts that correspond to the basic arithmetic operations on numbers; this section introduces addition, subtraction, negation, one form of multiplication, and—the key to them all—equality.

(1.4) ***Definition.*** Two matrices **A** and **B** are *equal* if and only if:
(a) **A** and **B** have the same number of rows and the same number of columns.
(b) All corresponding entries are equal—$\langle \mathbf{A} \rangle_{ij} = \langle \mathbf{B} \rangle_{ij}$ for all i and j.

This definition uses the phrase "if and only if," which you will see throughout this book. Recall that if P stands for some statement or condition and Q stands for another, then "P if and only if Q" means: if either of P and Q is true, then so is the other; if either is false, then so is the other. Like Siamese twins, P and Q always appear together.