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Exercises in Algebra: A Collection of Exercises in Algebra, Linear Algebra and Geometry

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EXERCISES IN ALGEBRA

A Collection of Exercises in Algebra, Linear Algebra and Geometry

Expanded English edition

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Preface to the Russian edition

Well presented collections of exercises for courses in higher algebra and linear algebra and geometry already exist (see, for example [5, 6]), and the publication of a new textbook of a similar nature needs to be justified. Changes in the above-mentioned courses at Moscow State University meant that new topics had been introduced, while the treatment of some other traditional topics had been shortened or even dropped altogether. As a result, university teachers had to use a large number of different books, of varying levels, when conducting seminars. In order to improve this situation, the professors of Higher Algebra at Moscow State University decided to prepare a new collection of exercises which would correspond to the modernized three-semester course.

The work assumed a collective nature from the very beginning. The author responsible for each chapter decided on the selection and amount of material, using criteria based on personal experience, whilst always attempting to moderate the quantity involved. In effect this approach has meant a lack of some standard numerical examples, and emphasis on the most noteworthy features. Thus the book contains the type of exercises actually offered in seminars. Almost all the sections contain exercises of a higher level of difficulty, for all of which hints are supplied. These exercises form the smaller part of each section, especially in the sections intended for the first semester. However their role increases in importance towards the end of the courses. The most difficult exercises may be presented and discussed in additional algebra seminars.

Exercises in Algebra was preceded by three rotaprint publications: *Basic Algebra, Linear Algebra and Geometry* and *Additional Chapters in Algebra* and the three parts of this book have similar contents to these. This arrangement of material follows the traditional structure of the lectures at the Department of Mechanics and Mathematics of the Moscow State University, and has been accepted in new curriculum planning at most universities in Russia. Of course the actual content of lectures and the order of exposition of material depends on the individual lecturer. Therefore it must be possible for textbooks and exercises to be used in a sufficiently flexible way. In any case, the authors have consciously allowed for some parallel and repeated material in different sections.

Theoretical comment is reduced to a minimum. However the material is arranged so that independent use of the book becomes increasingly important, especially in the final part. The theoretical basis of Parts One and Three can be found in [1] and that for Part Two in [2]. A significant number of exercises in this book have been taken from the collections of exercises mentioned in the references.

Lists of the definitions and symbols used, which may be helpful to the reader, are given at the end of the book. Any definition not listed will be found in the section 'Theoretical Material'. The latter contains the basic statements which are necessary for the completion of the exercises.

The authors wish to thank V.V. Batyrev, who improved the text. They are especially grateful to the professors of Higher Algebra and Number Theory at St. Petersburg University and to the professors of Algebra and Mathematical Logic at Kiev University, who have carefully reviewed the book and made many concrete suggestions.

The authors are grateful to Professor G.V. Dorofeev, who has given considerable attention to the principles used in ordering the content of the book and in standardizing the symbols, thus removing parallel material, as mentioned above, where it was excessive.

Postgraduate students have helped in checking the solutions and answers to the exercises. Nevertheless it is possible that a number of errors, proportional to the number of authors, remain, and we would be obliged if readers would give us their comments. We hope to be able to take into account diverse points of view and further improve the text.

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Foreword to the English edition

The main purpose behind the publication of *Exercises in Algebra*, as mentioned in the Preface, seems to have been achieved, judging by the comments received and by the experience of the authors — V.A. Artamonov, Y.A. Bahturin, E.S. Golod, V.A. Iskovskih, V.N. Latyshev, A.V. Mikhalev, A.P. Mishina, A.Y. Olshansky, A.A. Panchishkin, I.V. Proskuryakov, A.N. Rudakov, L.A. Skornyakov, A.L. Shmelkin, E.B. Vinberg — at Moscow State University. At the same time, some considerable defects were revealed, namely an insufficient number of computational exercises; a lack of series of typical exercises; inconvenient numbering of exercises and answers, and a too close juxtaposition of exercises of differing levels of difficulty.

The present edition, prepared mainly by V.A. Artamonov, and with the participation of virtually the same authors as the previous one, is aimed at removing these defects. The size of the book has been considerably increased, and not only because standard exercises have been added: special exercises, some of them fairly difficult, have also been included. These exercises, partially extracted from journals and monographs, can satisfy the demands of outstanding students and can help in the choice of topics for future research. They are located at the ends of some sections, after the symbol * * *.

The authors wish to thank E.V. Pankratiev and M.V. Kondratieva for preparing the camera-ready copy for the book.

We hope that the continuous numbering system will considerably simplify the use of the book. We look forward to a positive reaction from readers, and will gratefully consider all suggestions for elimination of any discrepancies which may have slipped in.

The authors are grateful to the Gordon and Breach Publishing Group for their willingness to publish *Exercises in Algebra* in the same series as *Linear Algebra and Geometry*. The education of mathematics students is usually based not only on lectures but also on seminars where students have the opportunity to discuss and solve exercises. This principle seems to apply in both the East and the West. It should be mentioned, as has been noted before, that: 'every textbook is written taking into account the traditions in a given university or, more generally, in the universities of a given country. My algebra textbook is no exception. At the same time, the exchange of ideas in the area of mathematics teaching in different countries is no less important than the exchange of ideas in research'[1].

This book was typeset using the AMS-TEX macro software package.

A.I. Kostrikin

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Contents

PART ONE
FOUNDATIONS OF ALGEBRA

CHAPTER 1

Sets and maps

1 Operations on subsets. Calculation of the number of elements

101. Let A_i $(i \in I)$, B be subsets of X. Prove that:

a) $$\left(\bigcup_{i \in I} A_i\right) \cap B = \bigcup_{i \in I} (A_i \cap B);$$

b) $$\left(\bigcap_{i \in I} A_i\right) \cup B = \bigcap_{i \in I} (A_i \cup B);$$

c) $$\overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i};$$

d) $$\overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$

102. Let X be an arbitrary set, and 2^X be the set of all its subsets. Prove that the operation Δ of *symmetrical difference*

$$A \Delta B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$$

on the set 2^X has the following properties:

a) $A \Delta B = B \Delta A$;

b) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$;

c) $A \Delta \emptyset = A$;

d) for any subset $A \subset X$ there exists a subset $B \subset X$ such that $A \Delta B = \emptyset$;

e) $(A \Delta B) \cap C = (A \cap B) \Delta (A \cap C)$;

f) $A \triangle B = (A \cup B) \setminus (A \cap B)$;

g) $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

103. For finite sets $A_1, \ldots, A_n$ prove that

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \ldots$$

$$+ (-1)^{k-1} \sum_{1\le i_1<\cdots<i_k\le n} |A_{i_1} \cap \cdots \cap A_{i_k}| + \cdots + (-1)^{n-1}|A_1 \cap \cdots \cap A_n|.$$

104. Prove that for any integer $n > 1$

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\ldots\left(1 - \frac{1}{p_r}\right)$$

where $p_1, p_2, \ldots, p_r$ are all different prime divisors of n, and $\phi(n)$ is the *Euler function.*

105. What is the maximal number of subsets which can be obtained starting from the given n subsets of a fixed set via the operations of intersection, union and complement?

106. Let $A,\ B,\ C$ be subsets of a set. Prove that $A \cap B \subseteq C$ if and only if $A \subseteq \overline{B} \cup C$.

2 Calculation of the number of maps and of the number of subsets. Binomial coefficients

201. Let X be a set of people in a room, Y be a set of chairs in this room. Suppose that

a) each chair is associated with a person who is sitting on it;

b) each person is associated with a chair on which he is sitting.

In what cases do a) and b) define a map $X \to Y$ and $Y \to X$? In which cases are these maps injective, surjective or bijective?

202. Prove that for an infinite set X and a finite subset Y there exists a bijective map $X \setminus Y \to X$.

203. Let $f : X \to Y$ be a map. The map $g : Y \to X$ is *left* (*right*) *inverse* for f, if $g \circ f = 1|_X$ ($f \circ g = 1_Y$, respectively). Prove that

a) a map f is injective if and only if it has left inverse;

b) the map f is surjective if and only if it has right inverse.

204. Establish a bijective correspondence between the family of all mappings from a set X into the set $\{0, 1\}$ and a set 2^X (see 102). Calculate $|2^X|$, if $|X| = n$.

205. Let $|X| = m$, $|Y| = n$. Find the number of all

a) maps

b) injective maps

c) bijective maps

d) surjective maps

from the set X into the set Y.

206. Let $|X| = n$. Find the number $\binom{n}{m}$ of all subsets of X of cardinality m.

207. Let $|X| = n$. Find the number of all subsets of X of even cardinalities.

208. Prove the *binomial formula of Newton:*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \qquad (n \in \mathbb{N}).$$

209. Let $|X| = n$ and $m_1 + \cdots + m_k = n$ $(m_i \geq 0)$. Find the number $\binom{n}{m_1, \ldots, m_k}$ of ordered partitions of X into k subsets containing respectively $m_1, \ldots, m_k$ elements.

210. Prove that:

a) $$(x_1 + \cdots + x_k)^n = \sum_{\substack{(m_1,\ldots,m_k) \\ m_1+\cdots+m_k=n,\ m_i \geq 0}} \binom{n}{m_1, \ldots, m_k} x_1^{m_1} \ldots x_k^{m_k};$$

b) $$\sum_{\substack{(m_1,\ldots,m_k) \\ m_1+\cdots+m_k=n,\ m_i \geq 0}} \binom{n}{m_1, \ldots, m_k} = k^n.$$

211. Prove that:

a) $$\binom{n}{m} = \binom{n}{n-m};$$

b) $$\sum_{i=0}^{n} \binom{n}{i} = 2^n;$$

c) $$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0;$$

d) $\sum_{i=1}^{n} i\binom{n}{i} = n2^{n-1};$

e) $\sum_{i=1}^{n}(-1)^i i\binom{n}{i} = 0,\ (n > 1);$

f) $\sum_{i=0}^{m}\binom{p}{i}\binom{q}{m-i} = \binom{p+q}{m};$

g) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},\ \ 1 \le k \le n;$

h) $\sum_{i=1}^{r}\binom{r+1}{i}(1^i + 2^i + \cdots + n^i) = (n+1)^{r+1} - (n+1);$

i) $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k+1};$

j) $\sum_{i=0}^{n}\frac{p(p+1)\dots(p+i-1)}{i!} = \frac{(p+1)\dots(p+n)}{n!};$

k) $\sum_{i=k}^{n-l}\binom{i}{k}\binom{n-i}{l} = \binom{n+1}{k+l+1},$ where $n \geqslant k + l \ge 0.$

212. Prove that $x^m + x^{-m}$ is a polynomial of degree m in $x + x^{-1}$.

213. Find the number of partitions of a number n into an ordered sum of k non-negative integers.

3 Permutations

301. Multiply the permutations in the indicated and in the inverse order:

a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 2 & 4 \end{pmatrix};$

b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 5 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 6 & 3 \end{pmatrix};$

c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 2 & 1 \end{pmatrix};$