MODEL BUILDING IN MATHEMATICAL PROGRAMMING

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To
Eileen, Anna and Alexander

Preface

Mathematical Programming is one of the most widely used techniques in Operational Research. In many cases its application has been so successful that its use has passed out of operational research departments to become an accepted routine planning tool. It is therefore rather surprising that comparatively little attention has been paid in the literature to the problems of formulating and building mathematical programming models or even deciding when such a model is applicable. Most published work has tended to be of two kinds. Firstly case studies of particular applications have been described in the operational research journals and journals relating to specific industries. Secondly research work on new algorithms for special classes of problems has provided much material for the more theoretical journals. This book attempts to fill the gap by, in Part 1, discussing the general principles of model building in Mathematical Programming. In Part 2, twenty practical problems are presented to which Mathematical Programming can be applied. By simplifying the problems much of the tedious institutional detail of case studies is avoided. It is hoped, however, that the essence of the problems is preserved and easily understood. Finally in Parts 3 and 4 suggested formulations and solutions to the problems are given.

Many books already exist on Mathematical Programming or, in particular, Linear Programming. Most such books adopt the conventional approach of paying a great deal of attention to algorithms. Since the algorithmic side has been so well and fully covered by other texts it is given much less attention in this book. The concentration here is much more on the building and interpreting of models rather than the solution process. Nevertheless it is hoped that this book may spur the reader to delve more deeply into the often challenging algorithmic side of the subject as well. It is, however, the author's contention that the practical problems and model building aspect should come first. This may then provide a motivation for finding out how to solve such models. Although desirable, knowledge of algorithms is no longer necessary if practical use is to be made of Mathematical Programming. The solution of practical models is now largely automated by the use of commercial package programs which are discussed in Chapter 2.

For the reader with some prior knowledge of Mathematical Programming, parts of this book may seem trivial and can be skipped or read quickly. Other parts are, however, rather more advanced and present fairly new material.

This is particularly true of the chapters on Integer Programming. Indeed this book can be treated in a non-sequential manner. There is much cross referencing to enable the reader to pass from one relevant section to another.

This book is aimed at three types of reader:

(1) It is intended to provide students in Universities and Polytechnics with a solid foundation in the principles of model building as well as the more mathematical, algorithmic side of the subject which is conventionally taught. For students who finally go on to use Mathematical Programming to solve real problems the model building aspect is probably the more important. The problems of Part 2 provide practical exercises in problem formulation. By formulating models and solving them with the aid of a computer a student learns the art of formulation in the most satisfying way possible. He can compare his numercial solution with that of other students obtained from differently built models. In this way he learns how to validate a model.

It is also hoped that these problems will be of use to research students seeking new algorithms for solving mathematical programming problems. Very often they have to rely on trivial or randomly generated models to test their computational procedures. Such models are far from typical of those found in the real world. Moreover they are one (or more) steps removed from practical situations. They therefore obscure the need for efficient formulations as well as algorithms.

(2) This book is also intended to provide managers with a fairly non-technical appreciation of the scope and limitations of Mathematical Programming. In addition by looking at the practical problems described in Part 2 they may recognize a situation in their own organization to which they had not realized Mathematical Programming could be applied.

(3) Finally, constructing a mathematical model of an organization provides one of the best methods of understanding that organization. It is hoped that the general reader will be able to use the principles described in this book to build mathematical models and therefore learn about the functioning of systems which purely verbal descriptions fail to explain. It has been the author's experience that the process of building a model of an organization can often be more beneficial even than the obtaining of a solution. A greater understanding of the complex interconnections between different facets of an organization is forced upon anybody who realistically attempts to model that organization.

Part 1 of this book describes the principles of building mathematical programming models and how they may arise in practice. In particular linear programming, integer programming and separable programming models are described. A discussion of the practical aspects of solving such models and a very full discussion of the interpretation of their solutions is included.

Part 2 presents each of the twenty practical problems in sufficient detail to enable the reader to build a mathematical programming model using the numerical data.

Part 3 discusses each problem in detail and presents a possible formulation as a mathematical programming model.

Part 4 gives the optimal solutions obtained from the formulations presented in Part 3. Some computational experience is also given in order to give the reader some feel for the computational difficulty of solving the particular type of model.

It is hoped that the reader will attempt to formulate and possibly solve the problems for himself before proceeding to Parts 3 and 4.

All the problems can be formulated in more than one way as mathematical programming models. Possibly some are better solved by means other than mathematical programming, but this is beyond the scope of this book. It will be obvious that some problems are sufficiently precisely defined that only one optimal solution should be expected in the sense of a maximum 'profit' or minimum 'cost' (although alternative optima may still occur as discussed in Chapter 6). For other problems, however, there may be no one correct answer. The process of modelling the practical situation into a mathematical programming format may necessitate approximations and assumptions by the modeller that result in different optimal solutions. How close such solutions are to those given in Part 4 may help to indicate the value or otherwise of mathematical programming for the type of problem being considered.

By presenting twenty problems from widely different contexts the power of the technique of mathematical programming in giving a method of tackling them all should be apparent. Some problems are intentionally 'unusual' in the hope that they may suggest the application of mathematical programming in rather novel areas.

Many references are given at the end of the book. The list is not intended to provide a complete bibliography of the vast number of case studies published. Many excellent case studies have been ignored. The list should, however, provide a representative sample which can be used as a starting point for a deeper search into the literature.

Many people have both knowingly and unknowingly helped in the preparation of this book by their suggestions and opinions. In particular I would like to thank my colleagues, while I was at Sussex University, Lewis Corner, Bernard Kemp, Pat Rivett, Steven Vajda and Will Watkins who have suggested problems and references to me. Also, through my chairmanship of the Mathematical Programming Study Group of the British Operational Research Society, I have had many informal conversations with other practicioners of Mathematical Programming which have proved of great value. I would like especially to acknowledge the motivation and ideas provided by Martin Beale, Tony Brearley, Colin Clayman, Martyn Jeffreys, Ailsa Land and Gautam Mitra. Finally I would like to thank Carol Kemp for her excellent typing of the manuscript.

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PART 1

CHAPTER 1

Introduction

1.1 The Concept of a Model

Many applications of science make use of *models*. The term 'model' is usually used for a structure which has been built purposely to exhibit features and characteristics of some other object. Generally only some of these features and characteristics will be retained in the model depending upon the use to which it is to be put. Sometimes such models are *concrete* as is a model aircraft used for wind tunnel experiments. More often in Operational Research we will be concerned with *abstract* models. These models will usually be *mathematical* in that algebraic symbolism will be used to mirror the internal relationships in the object (often an organization) being modelled. Our attention will mainly be confined to such mathematical models although the term 'model' is sometimes used more widely to include purely descriptive models.

The essential feature of a mathematical model in Operational Research is that it involves a set of *mathematical relationships* (such as equations, inequalities, logical dependencies, etc.) which correspond to some more down-to-earth relationships in the real world (such as technological relationships, physical laws, marketing constraints, etc.).

There are a number of motives for building such models:

(i) The actual exercise of building a model often reveals relationships which were not apparent to many people. As a result a greater understanding is achieved of the object being modelled.

(ii) Having built a model it is usually possible to analyse it mathematically to help suggest courses of action which might not otherwise be apparent.

(iii) Experimentation is possible with a model whereas it is often not possible or desirable to experiment with the object being modelled. It would clearly be politically difficult, as well as undesirable, to experiment with unconventional economic measures in a country if there was a high probability of disastrous failure. The pursuit of such courageous experiments would be more (though not perhaps totally) acceptable on a mathematical model.

It is important to realize that a model is really defined by the relationships which it incorporates. These relationships are, to a large extent, independent of the *data* in the model. A model may be used on many different occasions with differing data, e.g. costs, technological coefficients, resource availabilities, etc.

We would usually still think of it as the same model even though some coefficients had changed. This distinction is not, of course, total. Radical changes in the data would usually be thought of as a change in the relationships and therefore the model.

Many models used in Operational Research (and other areas such as Engineering and Economics) take standard forms. The Mathematical Programming type of model which we consider in this book is probably the most commonly used standard type of model. Other examples of some commonly used mathematical models are *simulation models*, *network planning models*, *econometric models*, and *time series models*. There are many other types of model all of which arise sufficiently often in practice to make them areas worthy of study in their own right. It should be emphasized, however, that any such list of standard types of model is unlikely to be exhaustive or exclusive. There are always practical situations which cannot be modelled in a standard way. The building, analysing and experimenting with such new types of model may still be a valuable activity. Often practical problems can be modelled in more than one standard way (as well as in non-standard ways). It has long been realized by operational research workers that the comparison and contrasting of results from different types of model can be extremely valuable.

Many misconceptions exist about the value of mathematical models, particularly when used for planning purposes. At one extreme there are people who deny that models have any value at all when put to such purposes. Their criticisms are often based on the impossibility of satisfactorily quantifying much of the required data, e.g. attaching a cost or utility to a social value. A less severe criticism surrounds the lack of precision of much of the data which may go into a mathematical model, e.g. if there is doubt surrounding 100 000 of the coefficients in a model how can we have any confidence in an answer it produces? The first of these criticisms is a difficult one to counter and has been tackled at much greater length by many defenders of cost-benefit analysis. It seems undeniable, however, that many decisions concerning unquantifiable concepts, however they are made, involve an implicit quantification which cannot be avoided. Making such a quantification explicit by incorporating it in a mathematical model seems more honest as well as scientific. The second criticism concerning accuracy of the data should be considered in relation to each specific model. Although many coefficients in a model may be inaccurate it is still possible that the structure of the model results in little inaccuracy in the solution. This subject is mentioned in depth in Section 6.3.

At the opposite extreme to the people who utter the above criticisms are those who place an almost metaphysical faith in a mathematical model for decision making (particularly if it involves using a computer). The quality of the answers which a model produces obviously depends on the accuracy of the structure and data of the model. For mathematical programming models the definition of the objective clearly affects the answer as well. Uncritical faith in a model is obviously unwarranted and dangerous. Such an attitude results from a total misconception of how a model should be used. To accept the first answer produced by a mathematical model without further analysis and questioning should be very rare. A model should be used as one of a number of tools for decision making. The answer which a model produces should be subjected to close scrutiny. If it represents an unacceptable operating plan then the reasons for unacceptability should be spelled out and if possible incorporated in a modified model. Should the answer be acceptable it might be wise only to regard it as an *option*. The specification of another objective function (in the case of a mathematical programming model) might result in a different option. By successive questioning of the answers and altering the model (or its objective) it should be possible to clarify the options available and obtain a greater understanding of what is possible.

1.2 Mathematical Programming Models

It should be pointed out immediately that *Mathematical Programming* is very different from *Computer Programming*. Mathematical Programming is 'programming' in the sense of 'planning'. As such it need have nothing to do with computers. The confusion over the use of the word 'programming' is widespread and unfortunate. Inevitably Mathematical Programming becomes involved with computing since practical problems almost always involve large quantities of data and arithmetic which can only reasonably be tackled by the calculating power of a computer. The correct relationship between computers and Mathematical Programming should, however, be understood.

The common feature which mathematical programming models have is that they all involve *optimization*. We wish to *maximize* something or *minimize* something. The quantity which we wish to maximize or minimize is known as an *objective function*. Unfortunately the realization that Mathematical Programming is concerned with optimizing an objective often leads people to summarily dismiss Mathematical Programming as being inapplicable in practical situations where there is no clear objective or there are a multiplicity of objectives. Such an attitude is often unwarranted since, as we shall see in Chapter 3, there is often value in optimizing some aspect of a model when in real life there is no clear cut single objective.

In this book we confine our attention to some special sorts of mathematical programming model. These can most easily be classified as *linear programming models*, *non-linear programming models*, and *integer programming models*. We begin by describing what a linear programming model is by means of a small example.

Example 1. *A Linear Programming* (*LP*) *Model* (*Product Mix*)

An engineering factory can produce 5 types of product (PROD 1, PROD 2, ..., PROD 5) by using two production processes: grinding and drilling.

After deducting raw material costs each unit of each product yields the