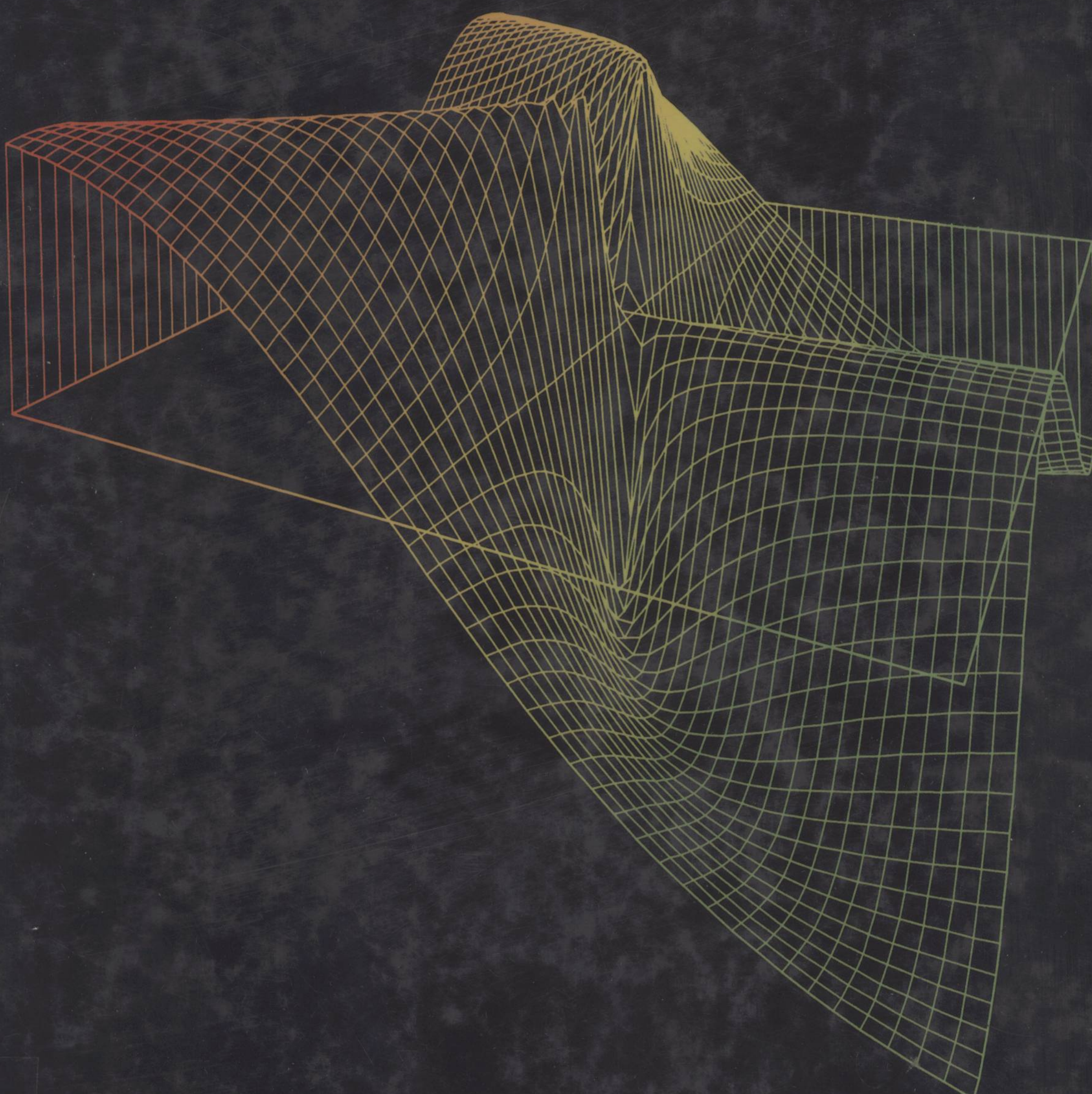
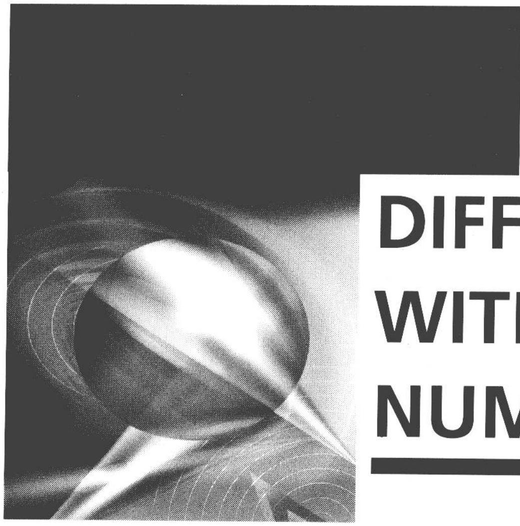
# DIFFERENTIAL EQUATIONS

## WITH GRAPHICAL AND NUMERICAL METHODS



BERNARD W. BANKS



# DIFFERENTIAL EQUATIONS WITH GRAPHICAL AND NUMERICAL METHODS



**BERNARD W. BANKS**

*California State Polytechnic University*



Prentice Hall

**PRENTICE HALL,** Upper Saddle River, New Jersey 07458

*Library of Congress Cataloging-in-Publication Data*

Banks, Bernard W.
Differential equations with graphical and numerical methods/Bernard W. Banks.
p. cm.
Includes index.
ISBN 0-13-084376-8
1. Differential equations.

QA371.B26 2001
515'.35--dc21 00-0022175
CIP

Acquisitions Editor: *George Lobell*
Production Editor: *Lynn Savino Wendel*
Assistant Vice President of Production and Manufacturing: *David W. Riccardi*
Executive Managing Editor: *Kathleen Schiaparelli*
Senior Managing Editor: *Linda Mihatov Behrens*
Manufacturing Buyer: *Alan Fischer*
Manufacturing Manager: *Trudy Pisciotti*
Director of Marketing: *John Tweeddale*
Marketing Manager: *Angela Battle*
Marketing Assistant: *Vince Jansen*
Associate Editor, Mathematics/Statistics Media: *Audra J. Walsh*
Editorial Assistant: *Gale Epps*
Art Director: *Ann France*
Interior Designer: *Marjory Dressler*
Cover Designer: *Jonathan Boylan*
Composition: *MacroTEX*
Chapter Opening Art: *Marjory Dressler*
Art Editor: *Grace Hazeldine*
Art Manager: *Gus Vibal*
Director of Creative Services: *Paul Belfanti*
Cover image credit: *Melvin L. Prueitt*



Printed in the United States of America
10 9 8 7 6 5 4 3 2 1

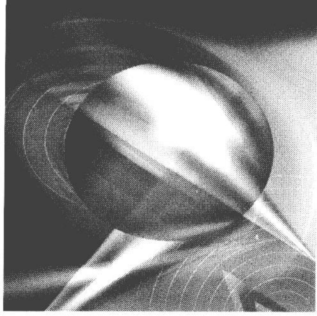
ISBN 0-13-084376-8

PRENTICE-HALL INTERNATIONAL (UK) LIMITED, LONDON
PRENTICE-HALL OF AUSTRALIA PTY. LIMITED, SYDNEY
PRENTICE-HALL CANADA, INC., TORONTO
PRENTICE-HALL HISPANOAMERICANA, S.A., MEXICO
PRENTICE-HALL OF INDIA PRIVATE LIMITED, NEW DELHI
PRENTICE-HALL OF JAPAN, INC., TOKYO
PEARSON EDUCATION ASIA PTE. LTD.
EDITORA PRENTICE-HALL DO BRASIL, LTDA., RIO DE JANEIRO

# DIFFERENTIAL EQUATIONS WITH GRAPHICAL AND NUMERICAL METHODS

*To my parents James and Phyllis Banks and my wife Harriet Lord.*

# PREFACE



Some time ago I searched for a textbook for a sophomore course in differential equations that would combine analytical (algebraic) methods of solution with graphical and numerical methods in a unified way. Some texts made computer graphics the center of the course and left out such topics as variation of parameters and infinite series. Other texts retained the traditional topics, but the graphics seemed to be grafted on as an afterthought. This book is an outgrowth of this failed search. The book retains almost all the traditional canon of differential equations, but it employs graphical and numerical methods from the outset, both as methods of solution and as means of illuminating concepts.

To employ graphical and numerical methods from the start, it was necessary to make first-order systems and reduction to first-order systems the focal point. First-order systems form the core subject matter of Chapters 1 through 5 and Chapter 7. Chapter 6 covers power series solutions, but even here first-order systems make a brief appearance in order to make clear why points at which the leading coefficient of a linear differential equation vanishes must be considered singular. Through first-order systems, solutions can easily be presented graphically with today's computer resources. This opens the way for visual interpretation of solutions and fields. First-order systems also provide the unified means of applying numerical methods to a very wide range of differential equations. Because of this, differential equations can be investigated that could not be considered in times gone by. Models of competing species, the pendulum, and the tunnel diode oscillator are taken up early in the text.

In spite of the emphasis on first-order systems, I have not neglected the basics of analytic solutions. Separable, linear, and exact equations are solved in the study of a single first-order equation in Chapter 2, and higher-order constant coefficient linear equations are treated in Chapter 4. However, the knowledge of first-order systems developed in Chapter 3 is used to establish the strategy for solving higher-order linear equations. Power series methods are also not neglected. Indeed, they cannot be, since they are needed in the solution of partial differential equations, which is the subject of Chapter 9.

Chapter 9 presents the solution of partial differential equations through the method of separation of variables and Fourier series. Chapter 10 introduces the reader to numerical methods of solution for partial differential equations. These two chapters were more difficult to write than the others because there is no unifying theme, such as first-order systems for ordinary differential equations. Nonetheless, graphics and numerical methods have been employed to help clarify ideas and to extend the range of equations solved. Computer algebra systems (CAS) such as Mathematica, Maple or MatLab (the Three M's) are used to advantage to illustrate convergence of Fourier series, graph modes of vibration for drumheads, and animate solutions. The chapter on numerical methods for partial differential equations is, I think, new in a book of this type. However, I believe it is entirely in keeping with the theme of this book and the availability of powerful computing resources. The use of a CAS makes the instability of some of the finite difference methods easy to explore, and it makes possible the exploration of some nonlinear partial differential equations.

Chapter 8 is a traditional treatment of the Laplace transform. The Laplace transform does not call for graphical or numerical methods, but I thought it important to include the Laplace transform because it is such an elegant way of dealing with constant coefficient linear equations and discontinuous forcing functions.

A large proportion of the exercises call for the use of a computer. The necessary software is available at the Prentice Hall web site:

`www.prenhall.com/banks`

## A Note to Instructors

For ordinary differential equations I have written a series of applications specific to the task at hand for the Windows and MacOs operating systems. These require no other support than the operating system. Packages for the entire book are also available at the web site for each of the Three M's. These routines are also listed completely in the instructors manual. If, however, you wish to have your students develop their own packages in one of the Three M's, Appendix D provides a guide to the development of these packages. This does have pedagogical value if you have the time. If you intend to cover the material on partial differential equations, then developing facility with one of the Three M's will be helpful.

The formal abstractions of vector spaces and linear transformations are introduced only in Section 9.4. Admittedly, this is quite late in the text, but it is not until this point that the references to linear combinations and linearity properties have sufficiently motivated the abstractions of a vector space and linear transformation. Together, Appendix A and Section 9.4 provide an introduction to the basic elements of linear algebra. They are self-contained, and Appendix A has exercises also. Thus Appendix A and Section 9.4 can be used at any time as supplementary lectures. If you feel the need to introduce concepts from linear algebra earlier, there is nothing to prevent you doing so. Indeed, if your students lack a good college algebra preparation, I recommend covering the material on solutions of systems of linear equations in Appendix A before going into Chapter 4.

The Supplementary Exercise sections at the end of chapters contain exercises that are of a more challenging nature or develop a topic that was not covered in the chapter. These exercises could be used as projects to be completed in, say, a ten-day period.

I confess to repeatedly abusing notation by referring to $f(x)$ as a function as well as the value of the function. I do this deliberately because I believe that using the precise notation for functions would tend to confuse students who are not yet mathematically very sophisticated. Besides, the precise notation can become quite cumbersome at times. Students will find "consider the function $f(x) = 3x$..." much more palatable than "consider the function $f : R \rightarrow R$ defined by $f(x) = 3x\ldots$."

It is intended that the book be covered in two semesters. The first semester should aim to cover Chapters 1 through 7 and the second Chapters 8 through 10. In one quarter, I have been able to cover Chapters 1 through 4 and Sections 6.1 and 6.2 with the omission of a few nonessential sections. There is no question that the material on partial differential equations is inherently more difficult than the rest of the text. However, this material is accessible if it is covered at a gentle pace. Sections that may be omitted without loss of continuity are 2.5, 3.5, 3.6, and all of Chapters 7, 8, and 10.

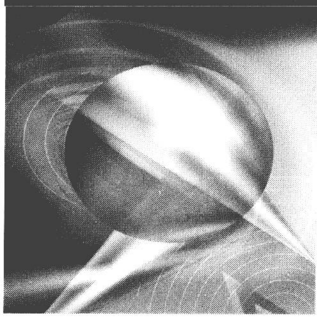
Finally, detailed suggestions for teaching sections of the text may be found in the instructors manual for the text. Also included are suggestions for laboratory activities, sample exam questions, and complete listings of the software packages.

## Acknowledgments

Several of my colleagues have taught from the text during its preparation. Their contributions have been invaluable. I would like to thank Dr. Weiqing Xie, Dr. Martin Nakashima, Dr. Alan Radnitz, and especially Dr. Harriet Lord for her constant help in the development of this text. I would also like to thank my editor, George Lobell, and his staff for their guidance and help.

*Bernard W. Banks*
*bwbanks@csupomona.edu*

# CONTENTS



## 5 First-Order Systems: Linear Methods 117



## 6 Series Methods and Famous Functions 166

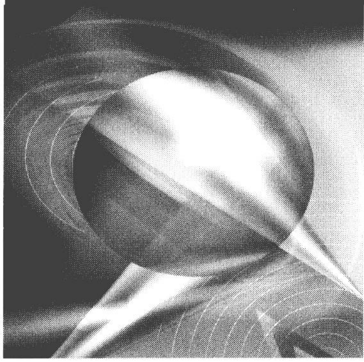


## 7 Bifurcations and Chaos 198



## 8 The Laplace Transform 230

CHAPTER 1



# INTRODUCTION

Differential equations arise in a wide range of practical problems. Whether you want to send an exploration mission to Mars or model the dynamics of populations of fish, differential equations will emerge as an essential part of the study of these tasks. The aim of this book is to provide you with an introduction to differential equations and the main ways in which differential equations can be studied. This book will describe what differential equations are, how they arise from practical problems, and the tools available for solving them.

This chapter introduces you to the general terms and concepts that are fundamental to the study of differential equations. Examples are given of practical problems that lead to differential equations. This will give you an introduction to the usefulness of the subject. Finally, the chapter closes with the description of a technique that reduces a wide range of differential equations to what can be considered a standard form. It is through this standard form that computing resources may be applied and important general results derived.

## 1.1 WHAT IS A DIFFERENTIAL EQUATION?

A differential equation may be described as an equation in which there are symbols for some of the derivatives of an unknown function. The order of such a differential equation is the order of the highest derivative in the equation. Differential equations come in two types, ordinary and partial. In an ordinary differential equation the unknown function is of one variable only, and the derivatives are ordinary. On the other hand, in a partial differential equation the unknown function is of several variables, and the derivatives are partial derivatives.

This book gives a brief introduction to the vast subject of partial differential equations, but the main focus is on equations of ordinary type. We shall delay discussing partial differential equations until the necessary groundwork in ordinary differential equations has been laid.

A **solution** to an ordinary differential equation is a function $\phi(x)$ defined on an interval $(a, b)$ such that, if the appropriate derivatives of $\phi(x)$ are substituted for the corresponding symbols for those derivatives in the equation, the equation becomes an identity for all $x \in (a, b)$.

We shall soon drop the adjective "ordinary" unless it is needed. So, unless otherwise stated, all differential equations may be assumed to be ordinary.

Here is an example of the ideas of a differential equation and a solution.

**Example 1.1.1** The following is a second-order ordinary differential equation.

$$y'' - y = x.$$

One solution (there are others) is

$$\phi(x) = -x + 2e^x.$$

Indeed, $\phi'(x) = -1 + 2e^x$, and $\phi''(x) = 2e^x$. Therefore,

$$\phi''(x) - \phi(x) = 2e^x - (-x + 2e^x) = x.$$

This is an identity for all $x$. In this case the interval of $x$ for which the equation holds is $(-\infty, \infty)$. ■

The square ■ will indicate the close of examples, definitions, and proofs.

Here is another example.

**Example 1.1.2** The function $\phi(x) = -1/(x - 1)$ is a solution to the equation

$$y' = y^2.$$

To verify this take the derivative and make the substitutions in the equation. This is the result.

$$\phi'(x) = \frac{1}{(x-1)^2} = \left(\frac{-1}{x-1}\right)^2 = (\phi(x))^2.$$

This is an identity for $x > 1$ or for $x < 1$. If we insist that the solution pass through (0, 1), then we choose the solution defined for $x < 1$. The point (0, 1) is called an initial condition, and we will have much more to say about this later. ■

## 1.2 APPLICATIONS OF DIFFERENTIAL EQUATIONS

Differential equations can be applied in many diverse areas. A very small sample of the kinds of applications that are possible is given next. At appropriate points in the text, applications will be made and evaluated in detail.

Science in general and physics in particular provide rich sources of differential equations. The reason for this, in physics at least, is that the laws of nature are most often expressed in terms of rates of change. For example Newton's second law relates the rate of change of momentum to force in the equation

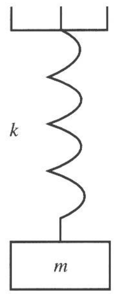



FIGURE 1.1

$$\frac{dmv}{dt} = F.$$

If the mass is a constant, we have the more familiar equation

$$ma = F,$$

where $a$ is the acceleration. You will recall that the acceleration is the second derivative of the position as a function of time. A specific example is given by a mass $m$ suspended from a spring (Figure 1.1).

Imagine that the mass is allowed to extend the spring until the force of gravity is balanced out by the spring. If the displacement of the mass is measured from this position, the force of gravity can be ignored. Let $x(t)$ be the displacement of the mass from the rest position, and assume that the spring produces a force that is proportional to the extension of the spring. Suppose that the formula is $F(x) = -kx$; then the second law gives us

$$mx''(t) = -kx(t)$$

or

$$mx''(t) + kx(t) = 0.$$

If we assume, in addition, there is a force of friction proportional to the velocity of the mass, then the equation of the motion is

$$mx''(t) = -cx'(t) - kx(t)$$

or

$$mx''(t) + cx'(t) + kx(t) = 0,$$

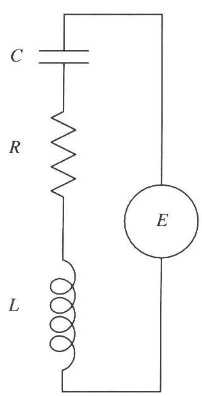



FIGURE 1.2

where $c > 0$ is the constant of proportionality. We assume this constant is positive because the force of friction is oppositely directed to the direction of motion.

The next example is from electronics. The circuit shown in Figure 1.2 has an inductor, resistor, capacitor, and generator that produces an electromotive force $E(t)$, which is a function of time. Kirchhoff's second law of electricity states that the sum of the voltage drops around the circuit must add up to the electromotive force. If $q(t)$ is the charge on the capacitor at time $t$, then the voltage drop across the capacitor is $q(t)/C$. Here the units for $C$ are faradays and the units for $q(t)$ are coulombs. The current is the rate at which charge passes a point in the wire. Thus $q'(t)$ is the current, and the units are amperes (coulombs per second). Resistance is measured in ohms, so if $R$ is the resistance, then the voltage drop across the resistor is $Rq'(t)$. Finally, the voltage drop across the inductor is proportional to the rate of change of the current. This proportionality constant is the inductance, $L$, and it is measured in henrys. Thus the voltage drop across the inductor is $Lq''(t)$. Therefore, adding the voltage drops we get

$$Lq''(t) + Rq'(t) + q(t)/C = E(t).$$

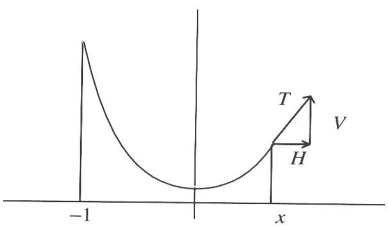


**FIGURE 1.3**

Here is yet another example from physics, this one from statics. If a cable hangs from two pins, it assumes a sort of parabolic shape. However, it is not a parabola. What is this shape? Consider Figure 1.3. The figure shows a hanging cable with part of the cable removed and a balance of forces diagram. The cable is hung from two pins at $(-1, 10)$ and $(1, 10)$. At $x$ the remaining cable to $(1, 10)$ has been removed and replaced with the tension force $T$ that the removed cable exerted on the cable to the left of $x$. Since the cable is flexible, this tension force is tangent to the curve at $x$, and the force has been resolved into the horizontal and vertical components of magnitudes $H$ and $V$, respectively. The cable is still in static equilibrium.

Since the cable is in static equilibrium, the tension forces in the cable are constant with respect to time. Consider the piece of cable from the bottom of the curve to the point $x$. The tension force $T_0$ at $x = 0$ pulls on the piece of cable from 0 to $x$ horizontally and to the left. Thus there is no vertical force at 0. Therefore, the vertical force, $V$, must equal the weight of this section of the cable in order for it to remain motionless. Thus

$$V = \int_0^x \rho\sqrt{1 + (y')^2}\,dt,$$

where $\rho$ is the weight density of the cable per unit of length, and $y(t)$ is the function whose graph is the hanging cable.

Remember that the tension force $T_0$ on the left of the cable segment is horizontal; so to maintain the equilibrium of the segment, $H = T_0$. Thus $H$ does not depend on $x$. Let $k$ be the reciprocal of the constant $H$.

The differential equation that $y$ must satisfy is found as follows. Note that, from the force triangle, $y'(x) = V/H$. Therefore,

$$y'(x) = \frac{V}{H} = kV = k\int_0^x \rho\sqrt{1 + (y')^2}\,dt.$$

Differentiating gives

$$y''(x) = k\rho\sqrt{1 + (y'(x))^2}.$$

The solution of this differential equation is the curve in which the cable hangs. Solving this equation is required in Supplementary Exercise 5 of Chapter 2.

In biology, a population of bacteria may reasonably be assumed to grow at a rate proportional to the population, provided the population has sufficient space and nutrients and no factors such as antibiotics are introduced. If $P(t)$ is the number of bacteria in the population at time $t$, then the differential equation is

$$P'(t) = rP(t).$$

Figure 1.4 shows a portion of a ski area and the corresponding contour map. Running across the middle of the region is a ridge, and just to the "northeast" of that there is a long valley. Recall that contours, or level curves, are curves of constant height on the terrain. Suppose that you wish to ski from the ridge into the valley by the fastest route. What path should you take? You should take the path that is always pointing downhill in the steepest direction, and this means that the path must cut the contours at right angles.

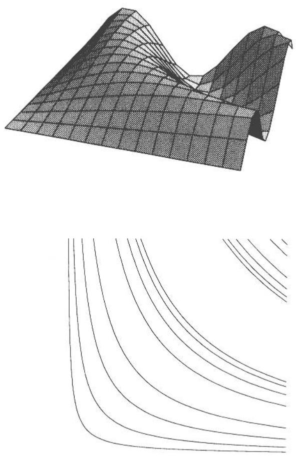


**FIGURE 1.4**

Suppose that the altitude of the mountain over the point $(x, y)$ is given by $z = f(x, y)$. Then a contour is a curve satisfying

$$f(x, y) = c.$$

You will recall that the gradient vector, defined by

$$\frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath},$$

is at right angles to the contour and is the direction of most rapid increase in height. Therefore, to ski down the mountain most rapidly, the skier must always be heading in the direction opposite to the gradient. Suppose also that the path down the mountain has been parameterized by the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$. Then

$$z(t) = f(x(t), y(t)),$$

and so it is $x = x(t)$ and $y = y(t)$ that we need to find. Skiing in a direction opposite to the gradient means that the horizontal velocity

$$\vec{v}(t) = x'(t)\vec{\imath} + y'(t)\vec{\jmath}$$

must be in the direction

$$-\frac{\partial f}{\partial x}\vec{\imath} - \frac{\partial f}{\partial y}\vec{\jmath}$$

at $(x(t), y(t))$. If we assume the parameterization is such that the velocity is actually equal to the negative of the gradient, this will be enough to describe the path down the mountain, although it will not necessarily give the correct speed for the skier. After all, the same path down the mountain can be skied making a snow plow all the way or skiing as fast as possible.