LECTURE NOTES IN PHYSICS

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Elements of Numerical Relativity

From Einstein's Equations to Black Hole Simulations

Springer

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To Montse, my dear wife and friend

Para mis padres, Francisco y Manuela,
que me enseñaron lo mas importante.
Y para mi amigo Jose y mi querida Eugenia,
que no han dejado que lo olvidara

Preface

We became involved with numerical relativity under very different circumstances. For one of us (C.B.) it dates back to about 1987, when the current Laser-Interferometer Gravitational Wave Observatories were just promising proposals. It was during a visit to Paris, at the Institut Henri Poincaré, where some colleagues were pushing the VIRGO proposal with such a contagious enthusiasm that I actually decided to reorient my career. The goal was to be ready, armed with a reliable numerical code, when the first detection data would arrive.

Allowing for my experience with the 3+1 formalism at that time, I started working on singularity-avoidant gauge conditions. Soon, I became interested in hyperbolic evolution formalisms. When trying to get some practical applications, I turned to numerical algorithms (a really big step for a theoretically oriented guy) and black hole initial data. More recently, I became interested in boundary conditions and, closing the circle, again in gauge conditions. The problem is that a reliable code needs all these ingredients to be working fine at the same time. It is like an orchestra, where strings, woodwinds, brass and percussion must play together in a harmonic way: a violin virtuoso, no matter how good, cannot play Vivaldi's *Four Seasons* by himself.

During that time, I have had many Ph.D. students. The most recent one is the other of us (C.P.). All of them started with some specific topic, but they needed a basic knowledge of all the remaining ones: you cannot work on the saxophone part unless you know what the bass is supposed to play at the same time.

This is where this book can be of a great help. Imagine a beginning graduate student armed only with a home PC. Imagine that the objective is to build a working numerical code for simple black-hole applications. This book should first provide him or her with a basic insight into the most relevant aspects of numerical relativity. But this is not enough; the book should also provide reliable and compatible choices for every component: evolution system, gauge, initial and boundary conditions, even for numerical algorithms.

This pragmatic orientation may cause this book to be seen as biased. But the idea was not to produce a compendium of the excellent work that has been made in numerical relativity during these years. The idea is rather to present a well-founded and convenient way for a beginner to get into the field. He or she will quickly discover everything else.

The structure of the book reflects the peculiarities of numerical relativity research:

- It is strongly rooted in theory. Einstein's relativity is a general-covariant theory. This means that we are building at the same time the solution and the coordinate system, a unique fact among physical theories. This point is stressed in the first chapter, which could be omitted by more experienced readers.
- It turns the theory upside down. General covariance implies that no specific coordinate is more special than the others, at least not a priori. But this is at odds with the way humans and computers usually model things: as functions (of space) that evolve in time. The second chapter is devoted to the evolution (or 3+1) formalism, which reconciles general relativity with our everyday perception of reality, in which time plays such a distinct role.
- It is a fertile domain, even from the theoretical point of view. The structure of Einstein's equations allows many ways of building well-posed evolution formalisms. Chapter 3 is devoted to those which are of first order in time but second order in space. Chapter 4 is devoted instead to those which are of first order both in time and in space. In both cases, suitable numerical algorithms are provided, although the most advanced ones apply mainly to the fully first order case.
- It is challenging. The last sections of Chaps. 5 and 6 contain front-edge developments on constraint-preserving boundary conditions and gauge pathologies, respectively. These are very active research topics, where new developments will soon improve on the ones presented here. The prudent reader is encouraged to look for updates of these front-edge areas in the current scientific literature.

A final word. Numerical relativity is not a matter of brute force. Just a PC, not a supercomputer, is required to perform the tests and applications proposed here. Numerical relativity is instead a matter of insight. Let wisdom be with you.

Palma de Mallorca,
April 2005

Carles Bona
Carlos Palenzuela Luque

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The Four-Dimensional Spacetime

1.1 Spacetime Geometry

Physics theories are made by building mathematical models that correspond to physical systems. General Relativity, the physical theory of Gravitation, models spacetime in a geometrical way: as a four-dimensional manifold. The concept of manifold is just a generalization to the multidimensional case of the usual concept of a two-dimensional surface. This will allow us to apply the well known tools of differential geometry, the branch of mathematics which describes surfaces, to the study of spacetime geometry.

An extra complication comes from the fact that General Relativity laws are formulated in a completely general coordinate system (that is where the name of 'General' Relativity comes from). Special Relativity, instead, makes use of inertial reference frames, where the formulation of the physical laws is greatly simplified. This means that one has to learn how to distinguish between the genuine features of spacetime geometry and the misleading effects coming from arbitrary choices of the coordinate system. This is why the curvature tensor will play a central role, as we will see in what follows.

1.1.1 The Metric

We know from differential geometry that the most basic object in the spacetime geometrical description is the line element. In the case of surfaces, the line element tells us the length dl corresponding to an infinitesimal displacement between two points, which can be related by an infinitesimal change of the local coordinates x^k in the surface. In the case of the spacetime, the concept of length has to be generalized in order to include also displacements in time (which is usually taken to be the 'zero' coordinate, $x^0 \equiv ct$). This generalization is known as the 'interval' ds, which can be expressed in local coordinates as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad (\mu, \nu = 0, 1, 2, 3) \, . \tag{1.1}$$

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We can easily see from (1.1) that the tensor $g_{\mu\nu}$ is going to play a central role. In the theory of surfaces, it has been usually called 'the first fundamental form'. In General Relativity it is more modestly called 'the metric' in order to emphasize its use as a tool to measure space and time intervals. The metric components can be displayed as a 4 by 4 matrix. This matrix is symmetric by construction (1.1), so that only 10 of the 16 coefficients are independent. Computing these 10 independent coefficients in a given spacetime domain is the goal of most Numerical Relativity calculations.

The metric tensor $g_{\mu\nu}$ is the basic field describing spacetime. One would need to introduce extra fields only if one wants to take into account non-gravitational interactions, like the electromagnetic or the hydrodynamical ones, but the gravitational interaction, as far as we know, can be fully described by the metric.

1.1.2 General Covariance

The most interesting property of the line element (1.1) is that it is invariant under generic (smooth) changes of the spacetime coordinates, namely

$$x^{\mu} = F^{\mu}(x^{\nu'}) \,. \tag{1.2}$$

This is because the values of space or time intervals are independent of the coordinate system one is using for labelling spacetime points. This means that the components of the metric must change in a suitable way in order to compensate the changes of the differential coefficients dx^{μ} in (1.1),

$$g_{\mu'\nu'} = g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \,. \tag{1.3}$$

We will say then that the metric transforms in a covariant way or, more briefly, that it behaves as a covariant tensor field under the general coordinate transformations (1.2).

The general covariance (1.3) of the metric means that, without altering the properties of spacetime, one can choose specific coordinate systems that enforce some interesting conditions on the metric coefficients. One can choose for instance any given (regular) spacetime point P and devise a coordinate system such that

$$g_{\mu'\nu'}\,|_{\mathrm{P}} = diag\{-c^2, +1, +1, +1\} \qquad \partial_{\rho'}\, g_{\mu'\nu'}\,|_{\mathrm{P}} = 0 \tag{1.4}$$

(local inertial coordinate system at P). This means that Special Relativity holds true locally (in the strongest sense: a single point at a time), and it will also be of great help in shortening some proofs by removing the complication of having to deal with arbitrary coordinate systems.

At this point, we must notice some ambiguity which affects to the very meaning of the term 'solution'. In the geometrical approach, one solution corresponds to one spacetime, so that metric coefficients that can be related by the covariant transformation (1.3) are supposed to describe the same metric, considered as an intrinsic tensor, independent of the coordinate system. In this sense, we can see how in exact solutions books (see for instance [1]) different forms of the same metric appear, as discovered by different authors. In the differential equations approach, however, the term solution applies to every set of metric components that actually verifies the field equations, even if there could be some symmetry (coordinate or 'gauge') transformation relating one of these 'solutions' to another.

This is by no way a mere philosophical distinction. If General Relativity has to be (as it is) general covariant, then the field equations must have two related properties:

- The equations must be unable to fully determine all the metric coefficients. Otherwise there would be no place for the four degrees of freedom corresponding to the general covariant coordinate (gauge) transformations (1.3).
- The equations must not prescribe any way of choosing the four spacetime coordinates. Otherwise there will be preferred coordinate systems and general covariance would be broken.

But in Numerical Relativity there is no way of getting a solution without computing the values of every metric component. This means that the differential system obtained from just the field equations is not complete, and one must prescribe suitable coordinate conditions before any numerical calculation can be made. The mathematical properties of the resulting complete system will of course depend of this choice of the coordinate gauge. We will come back to this point later.

1.1.3 Covariant Derivatives

The very concept of derivative intrinsically involves the comparison of field values at neighboring points. The prize one has to pay for using arbitrary coordinate systems is that one can no longer compare just field components at different points: one must also compensate for the changes of the coordinate basis when going from one point to another. In this way we can interpret the two contributions that arise when computing the covariant derivative of a vector field:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\ \rho\mu} v^\rho \, . \tag{1.5}$$

The first term corresponds to the ordinary partial derivatives of the field components, whereas the second one takes into account the variation of the coordinate basis used for computing these components. The Γ symbols in (1.5) are known as 'connection coefficients' because they actually allow to compare fields at neighboring points.

The covariant derivative of tensors with 'downstairs' indices contains connection terms with the opposite sign ('downstairs' components correspond to the dual basis). In the case of the metric, for instance, one has

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\ \rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\ \rho\nu} g_{\mu\sigma} \tag{1.6}$$

(notice that every additional index needs its own connection term).

The connection coefficients $\Gamma^\rho_{\ \mu\nu}$ are not tensor fields. They transform under a general coordinate transformation (1.2) in the following way:

$$\Gamma^{\rho'}_{\ \mu'\nu'} = \frac{\partial x^{\rho'}}{\partial x^\rho}\left[\Gamma^\rho_{\ \mu\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} + \frac{\partial^2 x^\rho}{\partial x^{\mu'}\partial x^{\nu'}}\right] . \tag{1.7}$$

The additional second derivatives terms appearing in (1.7) compensate exactly the analogous terms arising in the transformation of the partial derivative contributions in (1.5, 1.6), so that the covariant derivative of a tensor field is again a tensor field. Notice, however, that the extra second derivatives terms in (1.7) are symmetric in the lower indices. This means that the antisymmetric combinations

$$\Gamma^\rho_{\ [\mu\nu]} \equiv \frac{1}{2}\left(\Gamma^\rho_{\ \mu\nu} - \Gamma^\rho_{\ \nu\mu}\right) \tag{1.8}$$

correspond to the components of a tensor field (torsion tensor), because the antisymmetric part of the second derivatives terms in (1.7) actually vanishes.

Coming back to the metric tensor, the fact that the transformation of its first partial derivatives includes both first and second derivatives terms is the reason why one can define at any fixed point P the locally inertial coordinate system in such a way that both conditions in (1.4) hold true. It is natural to assume that the connection coefficients should also vanish in the local inertial system at P, in order to make sure that Special Relativity is fully recovered locally. These conditions imply that, in the local inertial coordinate system at P:

- The torsion (1.8) vanishes

$$\Gamma^\rho_{\ [\mu\nu]} = 0 . \tag{1.9}$$

- The metric is preserved by covariant differentiation

$$\nabla_\rho g_{\mu\nu} = 0 . \tag{1.10}$$

Notice that both (1.9) and (1.10) are tensor equations. And the vanishing of any tensor quantity in a local inertial system implies that it must actually vanish in any other coordinate system. This fact, allowing for (1.6), provides a very useful expression for the connection coefficients in terms of the first derivatives of the metric components:

$$\Gamma^\sigma_{\ \mu\nu} = \frac{1}{2} g^{\sigma\rho}[\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}] \tag{1.11}$$

(Christoffel symbols), where we have noted with 'upstairs' indices the components of the inverse matrix of the metric, namely

$$g^{\mu\rho} g_{\rho\nu} = \delta^{\mu}_{\ \nu} \ . \tag{1.12}$$

1.1.4 Curvature

Up to this point, all we have said could perfectly apply to the Special Relativity (Minkowski) spacetime. All the complications with covariant derivatives and connection coefficients could arise just from using non-inertial coordinate systems. Minkowski spacetime is said to be flat because a further specialization of the local inertial coordinate system can make the metric form (1.4) to apply for all spacetime points P simultaneously.

In General Relativity, in contrast, gravity is seen as the effect of spacetime curvature. So one must distinguish between the intrinsic effects of curvature (gravitation) and the sort of 'inertia forces' arising from weird choices of coordinate systems. Here again, this is a very well known problem from surface theory. The curvature of a surface can be represented by its curvature tensor (Riemann tensor, as it is known in General Relativity), which can be defined as follows:

$$\left(\nabla_\rho \nabla_\sigma - \nabla_\sigma \nabla_\rho\right) v^\mu = R^{\mu}_{\ \nu\rho\sigma} v^\nu \ , \tag{1.13}$$

so that it can be interpreted as a measure of the non-commutativity of (covariant) derivatives: a property that characterizes true curved spacetimes. The Riemann tensor $R^{\mu}_{\ \nu\rho\sigma}$ defined by (1.13) can be explicitly computed, allowing for (1.5), in terms of the connection coefficients:

$$R^{\mu}_{\ \nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\ \sigma\nu} - \partial_\sigma \Gamma^{\mu}_{\ \rho\nu} + \Gamma^{\mu}_{\ \rho\lambda} \Gamma^{\lambda}_{\ \sigma\nu} - \Gamma^{\mu}_{\ \sigma\lambda} \Gamma^{\lambda}_{\ \rho\nu} \ . \tag{1.14}$$

It is clear from (1.14) that in a flat spacetime, where there exists a coordinate system in which all connection coefficients vanish everywhere, the curvature tensor is zero, namely

$$R^{\mu}_{\ \nu\rho\sigma} = 0 \tag{1.15}$$

and, like any other tensor equation, it holds in any other coordinate system. Conversely, if the tensor condition (1.15) does not hold, then (1.14) tells us that there can not be any coordinate system in which all connection coefficients vanish everywhere and the manifold considered is not flat. It follows that (1.15) is a necessary and sufficient condition for a given spacetime to be flat. So finally we have one intrinsic and straightforward way to distinguish between genuine curved spaces and flat spaces 'disguised' in arbitrary coordinate systems.

1.1.5 Symmetries of the Curvature Tensor

Riemann curvature tensor is a four-index object. In four-dimensional spacetime, this could lead up to $4^4 = 256$ components. Of course there are algebraic symmetries that contribute to reduce the number of its independent components. Part of these symmetries can be directly obtained from the generic definition (1.14), which holds for arbitrary connection coefficients. The remaining ones come from taking into account the relationship (1.11) between the connection coefficients and the metric tensor. We have summarized them in Table 1.1.

Table 1.1. Algebraic symmetries of the Curvature tensor

Generic Case Symmetries	Metric Connection (1.11)
$R^{\mu}{}_{\nu\rho\sigma} = -R^{\mu}{}_{\nu\sigma\rho}$	$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$
$R^{\mu}{}_{\nu\rho\sigma} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\sigma\nu\rho} = 0$	$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$

But, even taking all these symmetries into account, one has still 20 algebraically independent components to deal with. One can easily realize, however, that lower rank tensors can be obtained by index contraction from the Riemann tensor. Allowing for the algebraic symmetries, there is only one independent way of contracting a pair of indices of the curvature tensor, namely

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} , \tag{1.16}$$

which is known as 'Ricci tensor' in General Relativity. It follows from the algebraic properties of the Riemann tensor that (1.16) is symmetric in its two indices, so it has only 10 independent components. Contracting again in the same way, one can get the Ricci scalar

$$R \equiv R^{\lambda}{}_{\lambda} = R^{\rho\sigma}{}_{\rho\sigma} . \tag{1.17}$$

The Ricci tensor (1.16) and the Ricci scalar (1.17) play a major role when trying to relate curvature with the energy content of spacetime. In three-dimensional manifolds, the Ricci tensor allows to obtain algebraically all the components of the curvature tensor (both of them have only six independent components). In the four-dimensional case this is no longer possible: the importance of the Ricci tensor comes instead from the Bianchi identities,

$$\nabla_{\lambda} R^{\mu}{}_{\nu\rho\sigma} + \nabla_{\rho} R^{\mu}{}_{\nu\sigma\lambda} + \nabla_{\sigma} R^{\mu}{}_{\nu\lambda\rho} = 0 , \tag{1.18}$$

which can be obtained directly from (1.14). One can contract two pairs of indices in (1.18) in order to get the following 'contracted Bianchi identity' for the Ricci tensor