Ramanujan/Thomas

INTERMEDIATE ANALYSIS

M. S. Ramanujan
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN

Edward S. Thomas
DEPARTMENT OF MATHEMATICS
STATE UNIVERSITY OF NEW YORK, ALBANY

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INTERMEDIATE ANALYSIS

A MATHEMATICS TEXTBOOK
UNDER THE EDITORSHIP OF
Carl B. Allendoerfer

Preface

This book began as a set of notes for a post-calculus course taken by prospective mathematics majors at the University of Michigan. The main objective of this course, and hence that of the book, is to provide the student with a feel for the way in which contemporary mathematicians design and build the machinery they use.

This approach dictates the initial subject matter of the book, set theory.

Every student of mathematics soon becomes aware of the fact that any mathematical concept can be stated ultimately in terms of sets, set operations, functions, and relations. Thus a knowledge of set theory is essential for the mathematician. Besides this, elementary set theory as presented here is simple enough, yet abstract enough, to provide the beginner with an abundance of problems on which to cut his teeth.

The more complicated set theoretic notions we have explored are used in that part of the book devoted to intermediate analysis. This portion includes a discussion of what the real number is, the elementary topology of the real line, and two chapters devoted to infinite series.

The reader who works through the text will have seen the development of mathematical notions from the extremely simple through the rather difficult.

We have adopted and stressed the viewpoint that rigor is essential in mathematics at each step in the development of an idea. That is, the idea must be rigorously defined and its consequences verified in every detail. On the other hand, once this point has been driven home, we expect the reader to supply many of the details himself. The majority of the exercises are designed to give the reader practice in supplying proofs and many of them fill gaps in the text.

Although the construction of the real numbers and verification of their properties are relegated to the appendixes, the reader is expected to use his intuitive knowledge of the real numbers throughout the text as a source of examples.

We have assumed, in several places, a knowledge of differential calculus—particularly in the chapters on infinite series. In addition to this there are several places in the text where the principle of induction is used. Finally, some exercises involving the concept of finiteness precede the definition of this term; these may be omitted without impairing the flow of the text.

We are indebted to several of our colleagues who read and commented on various portions of the manuscript and to many of the students who were subjected to preliminary versions of this book and who pointed out errors and omissions.

We acknowledge with thanks the aid we have received from the Department of Mathematics of the University of Michigan and, most especially, from the secretarial staff who typed much of the manuscript.

Finally, special thanks are due our wives for their encouragement during the writing and rewriting of this book.

M. S. R.
E. S. T.

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CHAPTER 0

Some Remarks About Logic

1. INTRODUCTION

This chapter is a brief, and by no means thorough, introduction to some of the rules by which the deductive process in mathematics is carried out. Almost everyone is aware of these rules, perhaps implicitly, but experience indicates that they may be forgotten or not believed just when needed most.

It is hoped that the remarks which follow will serve as a set of guidelines, not obstacles. The reader may have an easier time with this chapter if he realizes that it has little or no mathematical content of the type he recognizes. Rather it is metamathematical. We are merely going to make some remarks and observations about mathematics and, more specifically, about what a mathematical assertion is and what constitutes a proof. We also introduce some commonly used terminology and notation, much of which may already be part of the student's vocabulary.

2. ASSERTIONS

Here are three mathematical statements which we number for future reference.

2.1. If x and y are positive real numbers, then xy is positive.

2.2. If f is a differentiable function, then f is continuous.

2.3. If f is a continuous function, then f is differentiable.

Each of these statements has essentially the same form. We have one or more objects, denoted by letters (variables), under consideration and two statements about these objects, the hypothesis and the conclusion. Each statement is of the form "If (hypothesis), then (conclusion)." In statement 2.1, for example, the objects are two numbers, denoted by x and y. The hypothesis is that both numbers are positive and the conclusion is that their product is positive.

Each statement is an assertion to the effect that a certain hypothesis implies a certain conclusion in the sense that if any object (or set of objects) satisfies the condition stated as the hypothesis, then that object (or set of objects) also satisfies the condition stated as the conclusion. In this sense each of the "if, then" statements can be rewritten in the form "(hypothesis) implies (conclusion)." We call such statements *assertions* or *implications*.

A mathematical theory consists in large part of a body of valid assertions which are derived, via standard rules of logic, from basic definitions and axioms. Assertions which are considered especially important and/or "pretty" are given special titles, for example, *theorem*, *corollary*, *proposition*. Obviously, it is important to know what it means for an assertion to be valid (true) or invalid (false). We now discuss these concepts and touch on the related idea of a proof.

3. VALIDITY AND SOME REMARKS ON PROOFS

Let H and C be statements concerning one or more objects. We shall say that the assertion "H implies C" is *valid* or *true* provided that if an object or set of objects satisfies H, it also satisfies C; otherwise, we say the assertion is *invalid* or *false*.

Let us consider assertions 2.2 and 2.3 in light of the above definition. As everyone knows, assertion 2.2 is valid; your favorite calculus book contains a proof of it. It follows that since the function $f(x) = \sin x$ is differentiable, it is continuous. Similarly, $f(x) = e^x$ is differentiable, hence continuous. Indeed, once we know that 2.2 is valid, we can keep getting specific examples of continuous functions as long as our stock of differentiable ones holds out. One common mistake people make is to assume that the reverse of this process is true, that is, that an implication is valid if it is valid in a number of specific cases. The problem, of course, is that some cases not tested may show the assertion to be false. In connection with assertion 2.2, we also observe that the implication says nothing about the continuity of a function which is not differentiable; there are nondifferentiable functions which are continuous and ones which are not continuous.

Turning now to assertion 2.3, let us note that the functions $f(x) = \sin x$ and $f(x) = e^x$ satisfy both the hypothesis and the conclusion. By looking at these two functions, one might jump to the conclusion that assertion 2.3 is valid. However, it is not valid, because the absolute value function $f(x) = |x|$ is continuous, but, at the origin, it fails to be differentiable.

With these illustrations in mind, we list some observations about validity and proofs of assertions.

3.1. *To prove that "H implies C" is valid it does not suffice to exhibit examples of objects satisfying H which also satisfy C.*

3.2. *To prove that "H implies C" is invalid, it suffices to exhibit an object which satisfies H but not C.* This process is called finding a *counterexample*; thus $f(x) = |x|$ is a counterexample to assertion 2.3.

3.3a. *If "H implies C" is valid and an object does not satisfy C, then neither does it satisfy H.*

3.3b. Equally important, *to show that "H implies C" is valid, it suffices to show that if any object does not satisfy C, then neither does it satisfy H.*

Statement 3.3a is just a matter of juggling words in the definition of validity. Statement 3.3b can be illustrated by the following example. Suppose we wanted to prove assertion 2.2, "If f is a differentiable function, then f is continuous." We could go about this directly by showing that the condition for continuity (the δ-ε condition) follows from the condition for differentiability (the difference quotient condition). This would be what is usually called a *direct proof.*

There is another way to prove this assertion: We could show that if a function is not continuous, then it is not differentiable. Conceivably this would be done by writing down in terms of δ's and ε's what it means for f not to be continuous and, using this, to show that some difference quotient does not behave as it must in order for f to be differentiable. Proofs employing this tactic are often called *indirect proofs* and are quite common (although in the example chosen it so happens that a direct proof is neater).

There is a further refinement of the indirect proof, called *proof by contradiction.* The idea is this: To prove that "H implies C" we assume that there is an object which satisfies H but not C and, by some sort of argument, arrive at a conclusion which contradicts a known fact. The usual proof of the statement "If $x = \sqrt{2}$, then x is irrational," is an example of such a proof. One assumes that $\sqrt{2}$ is rational (that is, $\sqrt{2} = m/n$, where m and n are integers) and, by some tricky manipulations, shows that this contradicts some known algebraic facts.

In the above we have shown one way of not proving an assertion, given descriptions of the main varieties of proofs and discussed the idea of a counter example. We have not touched on the really crucial questions of what a proof is and how a person thinks one up. It requires a highly formalized language to give an accurate definition of the word "proof." We probably all agree that, intuitively, a proof of the validity of the implication "H implies C" is a logical chain of reasoning which starts with the statement that some object satisfies H and, using facts already proved or axioms, ends with the statement that the object also satisfies C. Strictly speaking, this is pretty much mumbo jumbo, since we do not know what a "logical chain of reasoning" is. Happily, it turns out that we can get by without a formal definition, because people involved in mathematics over a period of time tend to get by some sort of osmosis a feeling for what constitutes a mathematical proof.

Probably one learns to recognize whether an assertion has been proved or not by attempting to construct his own proofs and by criticizing those of others. This brings us to the second question. Proofs are concocted of a combination of experience, intuition, insight, ingenuity, and, sometimes, good luck. This book is designed to help the reader acquire some measure of the first three of these commodities; the last two are in great demand and each person must supply his own.

We close this section with a warning. It is easy to write down assertions which are meaningless, in the sense that the hypothesis or the conclusion contains variables which are not well-enough modified. The following example illustrates this: "If x is a real number, then $x + y = 10$."

The objects under consideration appear to be real numbers, but there is a variable, y, in the conclusion which is not quantified (modified) in the hypothesis. We have no way of proving or disproving this assertion without additional information about y.

For example, if we add to the hypothesis the requirement that $y = 5$ we get: "If x is a real number and $y = 5$, then $x + y = 10$." This is false, since $x = 2$ is a counterexample. But if we add the requirement that y denotes the quantity $10 - x$, we get a valid assertion: "If x is a real number and $y = 10 - x$, then $x + y = 10$."

The reader should assiduously avoid meaningless statements of the sort just described, although "dangling variables" may become harder to detect, as the level of abstraction increases.

4. NOTATION AND TERMINOLOGY

Instead of writing "If H, then C" or "H implies C," we frequently use the symbolism "$H \Rightarrow C$." Thus assertion 2.1 becomes: "x and y are positive real numbers $\Rightarrow xy$ is positive."

Consider now two statements, which we shall denote P and Q instead of H and C. The *converse* of the implication "$P \Rightarrow Q$" is the implication "$Q \Rightarrow P$"; thus the converse is obtained by reversing the roles of hypothesis and conclusion. Note that assertion 2.2 is the converse of assertion 2.3 and, of course, vice versa.

Obviously an assertion may be valid and its converse invalid. If, however, it happens that both "$P \Rightarrow Q$" and "$Q \Rightarrow P$" are valid, we say that P and Q are *equivalent* and write "$P \Leftrightarrow Q$." This last symbolism is frequently read "P if and only if Q"; here the "only if" corresponds to the arrow $\Rightarrow$ and the "if" to the arrow $\Leftarrow$.

For example, everyone knows that if x is a positive real number, then so is $x/10$ and, conversely, if $x/10$ is a positive real number, so is x. Thus we have "x is a positive real number if and only if $x/10$ is a positive real number."

Besides forming the converse, we can alter the assertion "$P \Rightarrow Q$" in other ways. One such way is to form the assertion "not $Q \Rightarrow$ not P", where "not P" and "not Q" are the negations, or denials, of the statements P and Q. The statement "not $Q \Rightarrow$ not P" is called the *contrapositive* of "$P \Rightarrow Q$." The contrapositive of assertion 2.2 is the assertion "If a function f is not continuous, then it is not differentiable."

The point we want to make is that *a given assertion and its contrapositive are equivalent* in the sense *that they are either both valid or both invalid.* In particular, to prove "$P \Rightarrow Q$" it suffices to prove "not $Q \Rightarrow$ not P." This is precisely the content of observation 3.3b and our remarks about indirect proofs in Section 3.

We close this chapter by establishing the following convention. *From now on* when we mean to say that the implication "$P \Rightarrow Q$"