SANSONE

ORTHOGONAL FUNCTIONS

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Revised English Edition

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Foreword

Modes come and go in Mathematics as in other fields of human endeavor. New modes first have a hard time breaking through until they are greeted with acclaim, copied and multiplied by the debutants, and then become oldfashioned and are disdained by the still younger set. In the meantime the old modes show great reluctance to disappearing, there are always some faithful souls who prefer the Paris mode of the twenties to, say the Princeton mode of the fifties. It is never safe to say that a particular field in Mathematics is dead or has outlived its usefulness. I have heard many pronouncements to that effect, often made with malice toward some; they have usually been belied by later events. A mathematician can and does choose the mathematics he prefers to do, not so the theoretical physicist who often cannot tell what type of mathematics his problem will lead to. We have seen over and over again during the last fifty years how one obsolescent mathematical theory after another had to be dug out of oblivion to meet the varying needs of physics. And there have always been young mathematicians willing to aid in the process who found rich reward in so doing.

The theory of orthogonal expansions had its origin in the debate concerning the vibrating string which animated the mathematical world two hundred years ago. The theory has had a place in the sun ever since, though naturally it meant different things to different times. How much it meant to the mathematicians of the eighteenth and nineteenth centuries can be read off from the monumental, 1800-page report on oscillating functions published by H. Burkhardt in 1908 covering the period 1727–1890. This was before the days of Fejér, Hardy, Hilbert, Lebesgue, Plancherel, F. and M. Riesz, Weyl, and Wiener, to mention only a few of the men whose work completely transformed the theory during the first third of this century. The new quantum mechanics found a rich storehouse to draw from; we mathematicians have been amply paid for the borrowings.

What is nowadays called the classical theory of orthogonal series is ably presented in G. Sansone's treatise which here appears in careful English translation. Since the term "classical" is often used in a derogatory sense by our vigorous youngsters—if lucky, their work will also become "classical"—I hasten to add that the term is here used as praise. The treatise contains what the student needs to know concerning the field, set in proper perspective, rigorously presented with due attention to detail and appropriate technique but nevertheless easy to read. It contains a wealth of information, factual and bibliographical, of use to the working analyst. If the reader should find the book short on maximal ideals, group characters, and other adjuncts of "modern" Fourier analysis, let that be a challenge to him to write a book in which these concepts are placed in the foreground. Such books are also needed.

American mathematics is deeply indebted to Professor Diamond for making this valuable text available to our students. Personally I am glad that my praise of the original had such beneficial effects. I can only hope that the translation will enjoy the same popularity in this country as the original has had and has in Italy.

EINAR HILLE

Preface

G. Vitali's monograph *Moderna Teoria delle Teoria delle Funzioni di Variabili Reale* finds here its natural sequel. The work should prove especially valuable to the applied mathematician who very often does not have access to the original papers. Its purpose is therefore to present general results and convenient criteria concerning Fourier series, Legendre series, Laguerre and Hermite polynomials.

Care has been taken to keep the necessary connections between this work and Vitali's monograph and it is hoped that the reader will be inspired to look further into a number of interesting questions concerning which only the essential points have been covered here.

The first chapter "Expansion in Series of Orthogonal Functions and Preliminary Notions on Hilbert Space" based on G. Vitali's *Geometria nello Spazio Hilbertiano,* contains the most pertinent results on expansion in series of orthogonal functions and includes some theorems on functions summable L^p and on convergence in the mean of order p.

The second chapter, "Expansion in Fourier Series" includes a treatment of the Gibbs phenomenon and a detailed discussion of the classical problem of Fourier on the distribution of heat in a plane. Also the validity of the theorems of Fejér and Lebesgue for (C, k) summability of Fourier series is extended to values of $k > 0$.

In the third chapter, "Expansion in Series of Legendre Polynomials," the representation of Legendre polynomials by the classical formulas of Mehler is established; the elegant expansion in series of Stieltjes-Neumann of $(1—x)^w$ is given prominence; and Bruns' inequalities for the zeros of $P_n(x)$ are established by the method of Szegö. In view of the great importance in applied mathematics of the expansion of functions in series of spherical harmonics in two variables, this chapter has been further amplified by a brief study of the spherical harmonics of Laplace and theorems on (C, k) and Poisson summability of such series.

Chapter four contains sufficient material to provide an introduction to the theory of representation of functions by series of Laguerre and Hermite polynomials. The reader who wishes to acquire a more complete knowledge of the theory is referred to G. Szegö's recent treatise *Orthogonal Polynomials* [107a] and the extensive bibliography of J. H. Shohat, E. Hille, and J. L. Walsh in their *Bibliography on Orthogonal Polynomials* [100b].

Formulas for the asymptotic approximation of Hermite polynomials are obtained by the method of Liouville-Stekloff and are used to obtain bounds on the orthogonal functions for complex arguments. The sixth article give a more precise form of Uspensky's formulas for the asymptotic approximation of the Laguerre polynomials.

GIOVANNI SANSONE

Translator's Note

In 1952 the third edition of Giovanni Sansone's treatise on Series Expansions of Orthogonal Functions appeared as Part II of *Moderna Teoria delle Funzioni di Variable Reale* in the series of Monografie di Matematica Applicata published by the Consiglio Nazionale delle Ricerche. Part I of this treatise was written by G. Vitali.

The enthusiastic reviews by Einar Hille of Sansone's work and the fact that much of the material, especially the chapters on series expansions in terms of Legendre polynomials and Laguerre and Hermite polynomials, was not readily available in English, suggested the desirability of translating the book.

The present volume comprises the first four chapters of Sansone's *Series Expansions of Orthogonal Functions*. The remaining two chapters, "Approximation and Interpolation" and "The Stieltjes Integral," which have no essential connection with the first four chapters, were omitted. Chapter II has been extended to include a section written by Sansone on the Fourier transform.

For the most part the translation is literal. The only essential departure from this procedure is in Chapter I, Sec. 5, "Convergence in the Mean," where the brief translator's note on the concept of internal convergence (*convergenza completa in media*) was added to the text.

In view of the numerous references in the Italian text to theorems in Part I of *Moderna teoria delle funzioni di variable reale*, it was decided to add an appendix listing those definitions and theorems from Part I to which references are made in the translation.

I am especially grateful to Professor Sansone for his encouragement and assistance in correcting the manuscript and selecting the material for the appendix. I also wish to acknowledge the contribution of the administration of Stevens Institute of Technology

in making available the time and secretarial assistance necessary to carry out the translation; the personal assistance of Frank Babina, one of my students at Stevens; the patient efforts of the secretary, Katherine Melis; and, finally, the encouragement and cooperation of Interscience Publishers, without which the present volume would not be possible.

AINSLEY H. DIAMOND

November, 1958.

Contents

CHAPTER I

Expansion in Series of Orthogonal Functions and Preliminary Notions of Hilbert Space

1. Square Integrable Functions

1. DEFINITION 1. A function will be said to have a property *almost everywhere* in a set g if it has that property for all points of g except for a set of measure zero. In future the abbreviation "a.e." will denote "almost everywhere."

DEFINITION 2. If g is a measurable set of points of a straight line, $f(t)$ is a function defined a.e. in g and $f^2(t)$ is integrable, then $f(t)$ is called *square integrable* in g.

THEOREM 1. If f_1 and f_2 are two functions measurable and square integrable in a set g, then the product $f_1 f_2$ is also integrable in g.

Proof. It is sufficient to show that $|f_1 f_2|$ is integrable in g. Clearly, $|f_1 f_2| \leqq \frac{1}{2}f_1^2 + \frac{1}{2}f_2^2$ a.e. in g and therefore $|f_1 f_2|$ admits a majorant which is integrable.

Henceforth, we shall tacitly assume that the functions under consideration are measurable.

COROLLARY. If f is a function square integrable in a set g of finite measure, then f is integrable in g.

Proof. In fact the constant 1 is square integrable in g, precisely because g has finite measure, and therefore the product $1 \cdot f$ is integrable in g.

THEOREM 2. If $f_1, f_2, \ldots, f_n$ are square integrable functions in g, and $c_1, c_2, \ldots, c_n$ are constants, then $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$ is a square integrable function in g.

Proof. $f_1, f_2, \ldots, f_n$ are finite a.e. in g, and since

$$(c_1 f_1 + c_2 f_2 + \ldots + c_n f_n)^2 = \sum_{i,k=1}^{n} c_i c_k f_i f_k$$

and the second member is integrable, it follows that the first member is integrable.

2. *Linearly Independent Functions*

1. DEFINITION 3. n functions $f_1, f_2, \ldots, f_n$ are called *linearly dependent* in g if there exist n constants $c_1, c_2, \ldots, c_n$, not all zero, for which the function $c_1 f_1 + c_2 f_2 + \ldots + c_n f_n$ is zero a.e. If such constants do not exist, the n functions are called *linearly independent*.

In a set g of measure zero, n functions are always linearly dependent. Accordingly, the discussion which follows is limited to sets of non-zero measure.

2. THEOREM 3. A necessary and sufficient condition for n functions

$$f_1, f_2, \ldots, f_n \tag{1}$$

square integrable in g, to be linearly dependent in g is the vanishing of Gram's determinant

$$G(f_1, f_2, \ldots, f_n) = \begin{vmatrix} \int_g f_1^2 dt, & \int_g f_1 f_2 dt, \ldots, & \int_g f_1 f_n dt \\ \int_g f_2 f_1 dt, & \int_g f_2^2 dt, \ldots, & \int_g f_2 f_n dt \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \int_g f_n f_1 dt, & \int_g f_n f_2 dt, \ldots, & \int_g f_n^2 dt \end{vmatrix}$$

(Gram [42]; Kowaleski [59]).

Proof. The condition is sufficient. In fact, if $G(f_1, f_2, \ldots, f_n)=0$ it is possible to find n constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero for which

$$\sum_{s=1}^{n} \lambda_s c_{r,s} = 0, \qquad (r = 1, 2, \ldots, n)$$

where

$$c_{r,s} = \int_g f_r f_s \, dt.$$

If F is defined by $F = \sum_{s=1}^{n} \lambda_s f_s$, then $\int_g F f_r \, dt = \sum_{s=1}^{n} \lambda_s c_{r,s} = 0$, whence

$$0 = \sum_{r=1}^{n} \lambda_r \int_g F f_r \, dt = \int_g \sum_{r=1}^{n} \lambda_r f_r \left(\sum_{s=1}^{n} \lambda_s f_s\right) dt = \int_g F^2 \, dt.$$

Now, since $\mu(g) \neq 0$ (where $\mu(g)$ denotes the measure of g), then F^2 and therefore F, is zero a.e. in g.

Conversely, if the functions (1) are linearly dependent in g, there will exist a system of constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ not all zero such that

$$\sum_{s=1}^{n} \lambda_s f_s = 0$$

a.e. in g. From this follows

$$0 = \int_g f_r \left(\sum_{s=1}^{n} \lambda_s f_s\right) dt = \sum_{s=1}^{n} \lambda_s \int_g f_r f_s \, dt = \sum_{s=1}^{n} \lambda_s c_{r,s} \quad (r = 1, 2, \ldots, n)$$

and therefore $G(f_1, f_2, \ldots, f_n) = 0$.

THEOREM 4. The rank of the matrix corresponding to the determinant $G(f_1, f_2, \ldots, f_n)$ gives the maximum number of linearly independent functions $f_1, f_2, \ldots, f_n$. If the rank is r, then r of the functions are linearly independent, and the other $n - r$ functions are linearly dependent on these.

Proof. Let r be the rank of the matrix correponding to the determinant G. Since G is symmetric it contains a non-vanishing principal minor of order r. Without loss of generality, we may suppose that this minor is formed from the first r rows and the first r columns of G. It follows that $G(f_1, f_2, \ldots, f_r) \neq 0$. Therefore the functions $f_1, f_2, \ldots, f_r$ are linearly independent.

From the fact that $G(f_1, f_2, \ldots, f_r, f_{r+j}) = 0$, $j = 1, 2, \ldots, n-r$, it follows from theorem 3 that every function f_{r+j} is linearly dependent on $f_1, f_2, \ldots, f_r$. In particular, we have

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_r f_r + \lambda_{r+j} f_{r+j} = 0$$

where $\lambda_{r+j} \neq 0$, and therefore f_{r+j} is a linear, homogenous combination of $f_1, f_2, \ldots, f_r$.

THEOREM 5. If the functions $f_1, f_2, \ldots, f_n$, square integrable in g, are linearly independent, then $G(f_1, f_2, \ldots, f_n) > 0$.

Proof. For n constants $c_1, c_2, \ldots, c_n$, not all zero $(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)^2 > 0$, a.e. in g. Therefore $\int_g (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)^2 dt > 0$, and finally the quadratic form in the $c_1, c_2, \ldots, c_n$

$$\sum_{i,k=1}^{n} c_i c_k \int_g f_i f_k \, dt$$

is positive definite, and its discriminant $G(f_1, f_2, \ldots, f_n) > 0$.

COROLLARY 1. If $f_1, f_2, \ldots, f_n$ are square integrable in g, then

$$G(f_1, f_2, \ldots, f_n) \geqq 0$$

and the equality sign holds only in case the given functions are linearly dependent in g.

COROLLARY 2. If f_1 and f_2 are two square integrable functions then

$$G(f_1, f_2) = \begin{vmatrix} \int_g f_1^2 \, dt, & \int_g f_1 f_2 \, dt \\ \int_g f_1 f_2 \, dt, & \int_g f_2^2 \, dt \end{vmatrix} = \int_g f_1^2 \, dt \int_g f_2^2 \, dt - \left(\int_g f_1 f_2 \, dt\right)^2 \geqq 0$$

and therefore

$$\left(\int_g f_1 f_2 \, dt\right)^2 \leqq \int_g f_1^2 \, dt \int_g f_2^2 \, dt.$$

The equality sign holds only in case f_1 and f_2 are linearly dependent in g. This inequality, usually referred to as the Schwarz inequality, was discovered independently by Bunikowsky in 1861 and Schwarz in 1885. The Schwarz inequality will arise as a particular case of the inequality which we shall establish in Sec. 9 of this chapter.

COROLLARY 3. If a function f defined in a set g of finite measure is square integrable, then

$$\left(\int_g f\,dt\right)^2 \leqq \mu(g) \int_g f^2\,dt.$$

Proof. Put $f_1 = f$, $f_2 = 1$ in the Schwarz inequality.

3. Elementary Notions of Hilbert Space

1. DEFINITION 4. We shall call a *real Hilbert Space,* or simply a *Hilbert Space,* or Space H, the set of all the functions which are square integrable in g. Two functions will be considered as the same element of H if they are equal almost everywhere in g. Any function whatever of the given set will be called a *point* of H; the function which vanishes almost everywhere will be called the *origin* of H.

Two points f_1 and f_2 are two distinct points of H if f_1 and f_2 are not equal almost everywhere.

2. DEFINITION 5. If f_1 and f_2 are two points of H, we shall call the *distance* between these two points the positive square root of $\int_g (f_1 - f_2)^2\,dt$; consequently if d is the distance between two points f_1 and f_2, then $d \geqq 0$ and

$$d^2 = \int_g (f_1 - f_2)^2\,dt.$$

The distance from a point f to the origin of H is given by $\sqrt{\int_g f^2\,dt}$.

3. DEFINITION 6. If f and φ are two distinct points of H and φ is distinct from the origin, we shall call the *straight line* passing through the points f and φ the totality of the points $f + \lambda\varphi$ where λ varies from $-\infty$ to ∞.

There is clearly a one-to-one correspondence between the points of the line and the values of λ.

Consider a line $\lambda\varphi$ passing through the origin; the points of this line which are a unit distance from the origin correspond to the values of λ for which $\lambda^2 \int_g \varphi^2\,dt = 1$; consequently, there exist on the line $\lambda\varphi$ two and only two points $\varphi/\sqrt{\int_g \varphi^2\,dt}$, and $-\varphi/\sqrt{\int_g \varphi^2\,dt}$ which are a unit distance from the origin. These two points will be called normal parameters of the line $\lambda\varphi$.

DEFINITION 7. A function φ square integrable in g will be called *normal* in g if

$$\int_g \varphi^2\, dt = 1.$$

The normal functions in g consist of all the points, and only those points, of H which are a unit distance from the origin.

To *normalize* a function φ square integrable and not vanishing a.e. in g, means to determine a factor c for which $c\varphi$ is normal in g, or, in other words, to find on the line $\lambda\varphi$ the points which are a unit distance from the origin. The required values of c are given by

$$c = 1/\sqrt{\int_g \varphi^2\, dt}, \qquad c = -1/\sqrt{\int_g \varphi^2\, dt}.$$

4. Let λf, $\mu\varphi$ be two lines through the origin; from the Schwarz inequality

$$\left(\int_g f\varphi\, dt\right)^2 \leqq \int_g f^2\, dt \int_g \varphi^2 dt$$

follows the existence of a number ω between 0 and π such that

$$\cos\omega = \left(\int_g f\varphi\, dt\right)/\sqrt{\int_g f^2\, dt \int_g \varphi^2\, dt}. \tag{1}$$

DEFINITION 8. The number ω between 0 and π which satisfies (1) will be called the *angle between the two lines* λf, $\mu\varphi$ in H.

In particular $\int_g f\varphi\, dt = 0$ is a necessary and sufficient condition for $\omega = \pi/2$.

We shall say in this case that the two lines λf, $\mu\varphi$ (or the two functions f and φ) are *orthogonal* in H (in g).

A necessary and sufficient condition for $\omega = 0, \pi$ is

$$\left(\int_g f\varphi\, dt\right)^2 = \int_g f^2\, dt \int_g \varphi^2\, dt,$$

which implies that the functions f and φ are linearly dependent (cf. Sec. 2) and the two lines λf, $\mu\varphi$ coincide.

It is easy to show that the functional metric defined in Sec. 3.2 has the properties of distance in ordinary space. For example, if d_1 and d_2 are the distances from the two points f_1