General Relativity and Matter

A Spinor Field Theory from Fermis to Light-Years

by

Mendel Sachs

Editorial Preface

There exist essentially two levels of investigation in theoretical physics. One is primarily *descriptive*, concentrating as it does on useful phenomenological approaches toward the most economical classifications of large classes of experimental data on particular phenomena. The other, whose thrust is *explanatory*, has as its aim the formulation of those underlying hypotheses and their mathematical representations that are capable of furnishing, via deductive analysis, predictions – constituting the particulars of universals (the asserted laws) – about the phenomena under consideration. The two principal disciplines of contemporary theoretical physics – quantum theory and the theory of relativity – fall basically into these respective categories.

General Relativity and Matter represents a bold attempt by its author to formulate, in as transparent and complete a way as possible, a fundamental theory of matter rooted in the theory of relativity – where the latter is viewed as providing an explanatory level of understanding for probing the fundamental nature of matter in domains ranging all the way from fermis and less to light years and more. We hasten to add that this assertion is not meant to imply that the author pretends with his theory to encompass all of physics or even a tiny part of the complete objective understanding of our accessible universe. But he does adopt the philosophy that underlying all natural phenomena there is a common conceptual basis, and then proceeds to investigate how far such a unified view can take us at its present stage of development. It is by persuasively arguing that indeed such an approach is able to lead us further than could previously considered superpositions of separate, disparate theories of matter, that Mendel Sachs' treatise makes a telling contribution to scientific thought.

With the Einstein Centenary still fresh in mind, the present volume is a timely tribute to the foremost thinker of our age. For, as Mendel Sachs points out, the view that the theory of relativity in its most general form should provide a basis for a fundamental theory of matter was one that Einstein himself pursued after his earlier work on special relativity had evolved into the theory of general relativity. It was, in fact, Einstein who said that the explanation of gravitational phenomena flowing from his field equations must not be regarded as more than a first step towards a general theory of matter. It is demonstrated in this monograph that such a comprehensive theory of matter ought in principle to incorporate all possible expressions of matter, which may be divided into force manifestations and inertial manifestations. With regard to the formal enunciation of the theory, an attack of this nature implies a single field theory that is consistent with the full extent of the algebraic symmetry group and the geometrical implications, in all areas, of general relativity. The symmetry group signifies in turn that the most primitive field variables capable of describing both the force and inertial aspects of matter must be sought in a spinor quaternion formalism for curved spacetime.

In Mendel Sachs' thorough-going discourse, the central axiom of general relativity – the principle of covariance – is augmented by two further assertions: the generalized Mach Principle (defined as a nonatomistic approach to matter in which all so-called intrinsic qualities of observed matter are viewed as being in actuality measures of coupling within a closed system) and the principle of correspondence. It is shown that the full exploitation of these three assertions yields a field theory of matter that is founded on a nonlinear, coupled, and self-consistent set of spinor field equations in curved spacetime. This analytical framework generates the inertial as well as the force manifestations of matter and encompasses the formal expression of quantum mechanics as a low-energy linear approximation for matter field equations that explicitly relate to the inertial manifestations of elementary matter. The author concludes this monograph by applying his theory to problems in elementary particle physics as well as in the astronomical and cosmological domains, indicating predictions in agreement with observational fact and offering suggestions for further avenues of investigation.

If, through his tenacious pursuit of the unifying power of relativity, Mendel Sachs has managed – as we believe he has – to deepen perceptibly our understanding of the fundamental behavior of matter, then *General Relativity and Matter* will have served the purposes of the *Fundamental Theories of Physics* series well indeed.

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Foreword

It is, perhaps, otiose to add a foreword to a book whose contents are so well described in its first chapter, but I welcome the opportunity to say something about a project which has linked together in a novel way two of the preoccupations of my professional life.

Over the 65 years since general relativity first emerged as a clear theory, basing its description of nature on differential geometry, the continuing successes and extensions of quantum mechanics have been a constant reproach to the pretensions of a view whose most certain tenet is the continuity of its description. They make it seem as if such a continuous description can never capture more than a fraction of nature; and this may well be so, but what Mendel Sachs urges in this book – and it is an argument that deserves very careful consideration – is that this fraction is significantly larger than has been thought, and that it includes a number of results that would usually be thought quantum mechanical. His thesis is, in short, that continuity is still alive and well, but of course something extra is needed.

Now there has always been a tradition of avoiding discontinuities in the mathematical description of nature. In its simplest form this can be seen in the description in Newtonian mechanics of a ball bouncing between two (perfectly elastic) walls. The velocity is evidently a step function of time but this causes no trouble since mechanics divides the motion into two parts, that between the walls which takes place under no forces according to Newton's first law and that at the walls, when a new empirical law (Newton's law of impact) or else a new empirical extension of the conservation of energy is introduced to describe the discontinuity. In recent years the detailed analysis of discontinuities, under the name of catastrophe theory, can be seen as carrying out the same kind of division but working from inside rather than outside the theory. But the line taken up by this book is quite different from that well-known approach. Certainly continuity is to be retained in the description of nature, but some further structure is to be introduced to carry the discreteness which we know to be inherent in atomic phenomena.

The great divide in this discussion came, historically, when Dirac formulated the wave equation for the electron, an equation which was Lorentz invariant although the quantities in it did not belong to any of the tensor representations of the orthogonal group. It did not matter that the mathematicians had been aware for years that the Lorentz group, amongst others, had two-valued representations. The shock to the community of physicists was as great as ever since they had been unaware of these other representations. Einstein and Mayer pondered over these new spin representations and found quaternion algebra emerging as a derived structure from the group representation theory. Eddington, in the same position, opted for the algebraic aspect of Dirac's work as the key to what had 'slipped through the net' and this led him to twenty years of speculative elaboration, years which seem at present to have led up a blind alley. But perhaps, as so often with Eddington, the elaboration of the details may be wrong when the underlying idea is magnificently right. As one reads through the present monograph it is possible to formulate what seems to be a much deeper idea: that algebraic structures in general and quaternion algebra in particular arise because they are the key to a basic discreteness.

It would be a bold man indeed who ventured to see in this idea all the richness of discrete description at present provided by quantum mechanics. But it is a wholly reasonable view to see some of this discreteness as available already in an algebraic reformulation of a continuous description, so that the way is cleared to see just how much discreteness is left to need a wholly novel quantum mechanical treatment. This approach is novel and so I will mention an analogy which is better known but which is quite different from anything that follows in the book and yet which partakes of the climate of ideas. Consider a one-dimensional classical conservative system, whose energy equation is

$$\frac{p^2}{2m} + V = E$$

and whose corresponding time-independent Schrödinger equation is therefore

$$\frac{h^2}{2m}\frac{d^2\psi}{dx^2} + (E - V)\psi = 0.$$

The conventional interpretation of quantum mechanics is that $|\psi|^2$ is proportional to the probability density. If, then,

$$\rho = |\psi|^2 = \psi\bar{\psi},$$

then

$$\frac{h^2}{2m}\rho'' = -2(E - V)\rho + \frac{h^2}{m}\psi'\bar{\psi}',$$

and hence

$$\frac{h^2}{2m}\rho''' = -4(E - V)\rho' + 2V'\rho.$$

If h were zero, ρ would therefore satisfy

$$\rho V' = 2\rho'(E - V),$$

and so $\rho \propto (1/\sqrt{E - V}) \propto (1/p)$. In the classical limit, then, the Schrödinger theory is telling us that the probability of finding the particle in any short interval is inversely proportional to the velocity with which it passes through that interval, which is classically exactly what we would expect. The point of this calculation is simply to emphasise that this probability distribution, which is rarely if ever considered classically, is automatically included in the Schrödinger equation, where it occurs mixed up with the uncertainty arising from the term $(h^2/2m)\rho'''$, although this latter uncertainty is what really interests us in quantum mechanics.

It is, I think, in the spirit of this example that the later part of Mendel Sachs's book should be read. The very striking numerical results for quantities which would normally be thought quantum mechanical may very well be produced because, with the help of a linear algebra, a classical theory can tell us more than we thought. Moreover conventional quantum mechanics has proved a very good calculational tool in many fields, but a much less satisfactory means of understanding them. If this book can improve understanding, it will have succeeded – whatever the fate of its numerical results.

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Preface

The primary aim of this monograph is to present the results of a research study that attempts to fully unify the logical and the mathematical aspects of the theory of general relativity. The approach is the one that I believe Einstein took in his later period. It views the theory of general relativity not as a theory only of the gravitational force manifestation of matter, but rather as a general theory of matter.

This approach proceeds by attempting to fully incorporate the Mach principle into the formalism so as to fuse the gravitational and electromagnetic force manifestations of matter with its inertial manifestations. With this approach, that I will argue is logically required by the axiomatic basis of the theory of general relativity, as a theory of matter, it will be shown that in accordance with Einstein's original motivation toward a unified field theory, generalization occurs that adds to the predictive capacity of the theory, incorporating the physics of the domain of elementary matter (of microscopic physics) with the domain of astronomy and cosmology; thus this approach unifies manifestations of matter from the domain of fermis (and less) to that of light-years (and greater) in terms of a nonsingular field theory of matter.

The view adopted here is that Einstein's theory of relativity is a single theory, not one part, called 'special relativity' that pertains uniquely to electrodynamics and high energy particle physics, and another part, called 'general relativity', that is to be taken uniquely as a theory pertaining to gravitation. It is, rather, a general field theory of matter, incorporating all possible force manifestations with the inertial manifestations.

In tracing the continuity from previous developments in theoretical physics, it should be noted at the outset that a significant feature of the nature of matter necessarily comes with the logical requirement of general relativity theory that it be based on the full use of the field concept – as Faraday had originally suggested to underlie the most general expressions of the laws of nature. But Faraday's notion of the 'field' is also generalized in Einstein's theory. For instead of starting with the potential field of force and its action on an (ambiguously defined) 'test body', the generalized field theory must incorporate the 'test particle' with the rest of the entire system, expressed in terms of an underlying continuous field that now describes a *closed system* from the outset.

The view that a part of this system plays the role of a 'test body', that 'looks at' its environment, does not emerge from the theory until the asymptotic limit is taken, in which a portion of the closed system approaches a sufficiently weakly coupled state in which *it appears, as a first approximation*, to be disconnected from its environment. Indeed, the difference between the asymptotic limit in the description of a closed system, in which a component seems to be almost uncoupled, on the one hand, and on the other hand, a completely uncoupled 'test body' that is slightly perturbed by its environment, yields some important contrasting mathematical and physical consequences.

Such incorporation of the 'test matter' with the rest of the (assumed) closed system, in terms of a single underlying 'field of interaction', rather than Faraday's potential field of force, is a generalization of the field concept that occurs in a natural way with the union of the Mach principle and Einstein's theory of relativity. This generalization will be continually emphasized throughout this book. It is especially pertinent in regard to the way in which this theory transcends the quantum mechanical theory of matter, as a field approach that corresponds asymptotically, in the linear limit, with the quantum mechanical (particle) theory of matter and its incorporation of the measurement process. Similarly, it is a theory that transcends Newton's theory of universal gravitation, though corresponding mathematically with the latter as a linear approximation under appropriate conditions. In both of these cases, in the elementary particle domain and the astronomical domain, the superseded theories are totally replaced, from the conceptual views, though the mathematical forms of the older theories are maintained as particular approximations in appropriate limits. The latter is a use of the *correspondence principle* in the structuring of Einstein's field theory. In both the elementary particle and astronomical domains the concepts of linearity, atomism and action-at-a-distance are replaced with the concepts of nonlinearity, the continuous (nonsingular) field concept and the finite speed of propagation of interaction between the coupled components of a material system.

The exploitation of these ideas leads to a general theory of matter – from fermis to light-years – by fully expressing the logic of space-time that is imposed by the theory of general relativity. This logic (as discussed in Chapter 1) is in two parts – algebraic and geometric. The algebraic part of the logic of space-time leads to the form of the irreducible representations of the underlying Lie group of relativity theory, which obey the rules of a quaternion algebra, and the basis functions of these representations, which are two-component spinor variables.

The geometrical part of the logic of space-time leads to a *curved* coordinate system in which the two-component spinor fields (and all other field variables of the theory) must be mapped. The latter space is characterized by a (generalized) Riemannian metric. It is shown (in Chapter 3) that in accordance with the symmetry requirements of general relativity theory – that the transformations are continuous and analytic – when the space-time reflection symmetry elements are removed from the invariance group of the ordinary Riemannian metric,

$ds^2 = g^{\mu\nu}(x)\,dx_\mu\,dx_\nu$, the latter *factorizes* into the product of a quaternion invariant metric and its conjugate, $ds\,\widetilde{ds}$, where $ds = q^\mu(x)\,dx_\mu$ and $q^\mu(x)$ is, geometrically, a four vector in the curved space-time, but each of its four components is a quaternion, depending on four real number components, rather than a single real number. The quaternion metrical field $q^\mu(x)$ is then a 16-component variable. Thus, the 10-component metrical field $g^{\mu\nu}(x)$ of Einstein's formulation – that includes the reflection symmetry elements in its underlying group – factorizes into a form proportional to $(q^\mu\tilde{q}^\nu + q^\nu\tilde{q}^\mu)$, in which the 16-component metrical field $q^\mu(x)$ does not entail reflection covariance.

Based on this mathematical background, it is shown (in Chapter 6) that all of the standard tensors of a Riemannian space-time may be re-expressed in terms of the quaternion metrical fields and their derivatives. From the Lagrangian density, in terms of the Riemann scalar curvature field (as a function of the quaternion variables), the variational principle leads to the metrical field equations in $q^\mu(x)$. The latter then appears as a *factorization* of Einstein's tensor field equations. The latter are 16 (rather than 10) field relations at each space-time point. The increased number of degrees of freedom in the metrical field follows from the removal of the reflection symmetry elements from the underlying (Einstein) group of general relativity theory – so as to yield the irreducible form of its representations.

It is shown that this 16-component metrical field incorporates the equivalent of the 10-component field of Einstein's original tensor formulation and a 6-component field that solves field equations of the form of Maxwell's equations for electromagnetism. Generalization also occurs in the appearance of a more general form of the geodesic equation in quaternion form, expressing proper time for a trajectory of the space-time as a 4-parameter set. This generalization leads to new astrophysical results (Chapter 7) pertaining to planetary motion, stellar dynamics in host galaxies, relating to their spiral structure, and the Hubble law.

It is shown (Chapter 4) how the quaternion metrical field appears in a crucial way in the matter field equations that represent elementary matter in the microscopic domain, leading to a derivation of the inertial manifestations of matter – in accordance with the predictions of the Mach principle. The matter field equations themselves are first order nonlinear differential-integral equations that *approach* the form of quantum mechanics *as a linear approximation*. Thus, the formal (Hilbert space) structure of quantum mechanics appears in this theory of elementary matter only as a linear (low energy) approximation for a general formalism that is based on the axioms of general relativity, rather than the quantum theory.

Summing up, the general approach to general relativity theory that is developed in this monograph, which is strictly in accordance with Einstein's views of a unified field theory, leads to a unification of the force manifestations of matter (thus far in terms of gravitation and electromagnetism) and its inertial manifestations. Such a unification is derived in this monograph in terms of a set of self-consistent, inter-dependent field equations. It is found that the extension of the unification of fields to include the inertial manifestations of matter was a crucial element in the achievement of such a unified field theory. It is also an extension that generalizes the (arbitrarily defined) 'rod-clock' expression of measurement, according to Einstein's initial discussions of special relativity theory, to a fully covariant field representation of matter, including the 'observer' component of an 'observer-observed' relation, where the 'observed' might be micro-matter.

It is noted, finally, that the incorporation of the 'measurement', when described in special cases with macro-variables, with the measured, when described by micro-variables, was a fundamental requirement of the Copenhagen school, that Einstein had not yet accomplished in general relativity theory. With the latter theory, as a basis for a fundamental understanding of matter, the variables of the 'observer' and the 'observed' must be introduced also, though symmetrically, as component fields that obey the same rules, while in Bohr's view of quantum mechanics, they are introduced, *necessarily*, asymmetrically, where the 'observer' component obeys the rules of classical physics while the 'observed' component obeys the rules of quantum physics.

In this monograph, then, we resolve one of the problems of matter that was addressed in the Einstein–Bohr debates, by completing Einstein's field theory of matter *and* by satisfying Bohr's requirement of incorporating the measured with the measurer, though in this relativistic field theory it is done in a fully covariant manner, with a unified field that explicitly incorporates the inertial manifestations of matter.

This monograph is divided into three parts: Concepts, Mathematical Preliminaries, and The Field Equations. These divisions, in turn, are divided among one chapter in Part I, two chapters in Part II: 'Vector-Tensor Analysis' and 'Spinor-Quaternion Analysis' and four chapters in Part III: 'The Matter Field Equations', 'The Electromagnetic Field Equations', 'The Gravitational Field Equations and Unification with Inertia and Electromagnetism', and 'Astrophysics and Cosmology'.

An extensive bibliography on general relativity theory will not be given in the text. On the general background material for the subject, I have found the following books most useful: *The Principle of Relativity* by Einstein and others (Methuen, 1923); *The Meaning of Relativity* by Einstein (Princeton, 1955), fifth edition; *Space-Time-Matter* by Weyl (Dover, 1922); *Space-Time Structure* by Schrödinger (Cambridge, 1954); *The Theory of Relativity* by Møller (Oxford, 1952); *The Theory of Relativity* by Pauli (Pergamon, 1958); *Introduction to General Relativity* by Adler, Bazin, and Schiffer (McGraw-Hill, 1975), second edition.

On Einstein's philosophical view of relativity theory, the 'Autobiographical Notes' in *Albert Einstein – Philosopher-Scientist* (Northwestern, 1949), edited by P. A. Schilpp, is strongly recommended. I feel that the best source for the background of the Mach principle is Mach's own work, *The Science of Mechanics* (latest reprinting, Open Court, 1960). I have discussed the general philosophy of relativity theory in my books: *Ideas of the Theory of Relativity* (Israel Universities Press, 1974) and *The Field Concept in Contemporary Science* (Thomas Publ., 1973).

On aspects of the mathematical background material I have found the following texts useful: Clear insight into differential geometry is given in *Riemannian Geometry* by Eisenhart (Princeton, 1950). An extremely good presentation of the application of variational calculus to physics is given in *The Variational Principles of Mechanics* by Lanczos (Toronto, 1962). On Group Theory, I have found the following texts most helpful: *Continuous Groups of Transformations* by Eisenhart (Princeton, 1933), *Topological Groups* by L. Pontrjagin (Princeton, 1958), *Group Theory and its Application to Quantum Mechanics of Atomic Spectra* by Wigner (Academic, 1959).

The conventional ideas of the quantum theory of measurement, from the view of its underlying logic and mathematics, are expressed most succinctly and rigorously by P. A. M. Dirac in his book, *The Principles of Quantum Mechanics* (Oxford, 1958), fourth edition.

A great deal of stress has been placed in this monograph on the role that must be played by spinor and quaternion variables as the primitive fields in relativity theory (whether expressed in the form of special or general relativity). Because there is not an abundant literature on this type of mathematical formalism, the algebra and calculus of spinor-quaternion variables in special and general relativity theory will be developed here from first principles (Chapter 3). Other reference to this subject must start, of course, with the original papers on quaternions by their discoverer, William Rowan Hamilton. These were published during the nineteenth century in the *Proceedings of the Royal Irish Academy*. A more accessible volume that collects these works is *The Mathematical Papers of Sir William Rowan Hamilton*, Vol. III, edited by Halberstam and Ingram (Cambridge, 1967).

Other references to related mathematical developments on spinor and quaternion variables, that I have benefited from are: H. S. Ruse, *Proc. Roy. Soc.* (*Edinburgh*) **57**, 97 (1937); C. W. Kilmister, *Proc. Roy. Soc.* (*London*) **A199**, 517 (1949); **A207**, 402 (1951); W. L. Bade and H. Jehle, *Rev. Mod. Phys.* **25**, 714 (1953); C. C. Chevalley, *The Algebraic Theory of Spinors* (Columbia, 1954); G. Szekeres, *J. Math. Mech.* **6**, 471 (1957); A. W. Conway, *Proc. Roy. Irish Academy* **A50**, 98 (1945); *Proc. Roy. Soc.* (*London*) **A162**, 145 (1937).

I have also greatly benefited from a series of lectures that were given by W. Pauli at the University of California in 1958. These were printed by the University of California Radiation Laboratory, entitled, 'Lectures on Continuous Groups and Reflections in Quantum Mechanics' (UCRL8213), edited by R. J. Riddell, Jr. (October, 1958).

This monograph is addressed primarily to researchers and other readers in theoretical physics, both at the graduate student and the more advanced physicist and mathematician levels. If the presentation leaves the reader with the feeling that further investigations into the continuum field approach to matter, according to the theory of general relativity, can be very much alive, exciting and perhaps even crucial to the future development in our understanding of the material world, then I would feel that one of the primary aims of this book would have been fulfilled.

The presentation of this book grew out of a graduate level course that I have been teaching since 1962, each academic year rewriting and adding to the manuscript as my research has progressed and as my own understanding has correspondingly evolved. Most of the lecturing on this subject has been at my home institution, State University of New York at Buffalo, and some of it at other universities and institutes where I have been a Visiting Professor, in the United States, Canada and abroad. Most sincere thanks are due to my students at my home institution and to the students and faculties of the institutions that I have visited, for their helpful comments and hospitalities.

The final draft of this book was written while I was on a sabbatical leave at the Weizmann Institute of Science in Rehovot, Israel. I am grateful to my hosts, the Department of Nuclear Physics, for their hospitality and for giving me the opportunity to spend 7 months in this very beautiful and inspiring country. It is my ardent hope that the following thought of one of the great leaders of Israel, inscribed at his final resting place at the Institute, will one day become a reality:

> "I feel sure that science will bring to this land both peace and a renewal of its youth creating here the springs of new spiritual and material life. And here I speak of science for its own sake and of applied science.
>
> Chaim Weizmann
> 1946"

I gratefully acknowledge permission from the Estate of Albert Einstein to quote from the writings of Albert Einstein and from the Cambridge University Press to quote from the writings of Albert Einstein and William Rowan Hamilton that appear in their publications.

Last, but not least, I wish to express my most grateful appreciation to my wife, Yetty, for her extreme patience and encouragement; without these this book would have never come to fruition.

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